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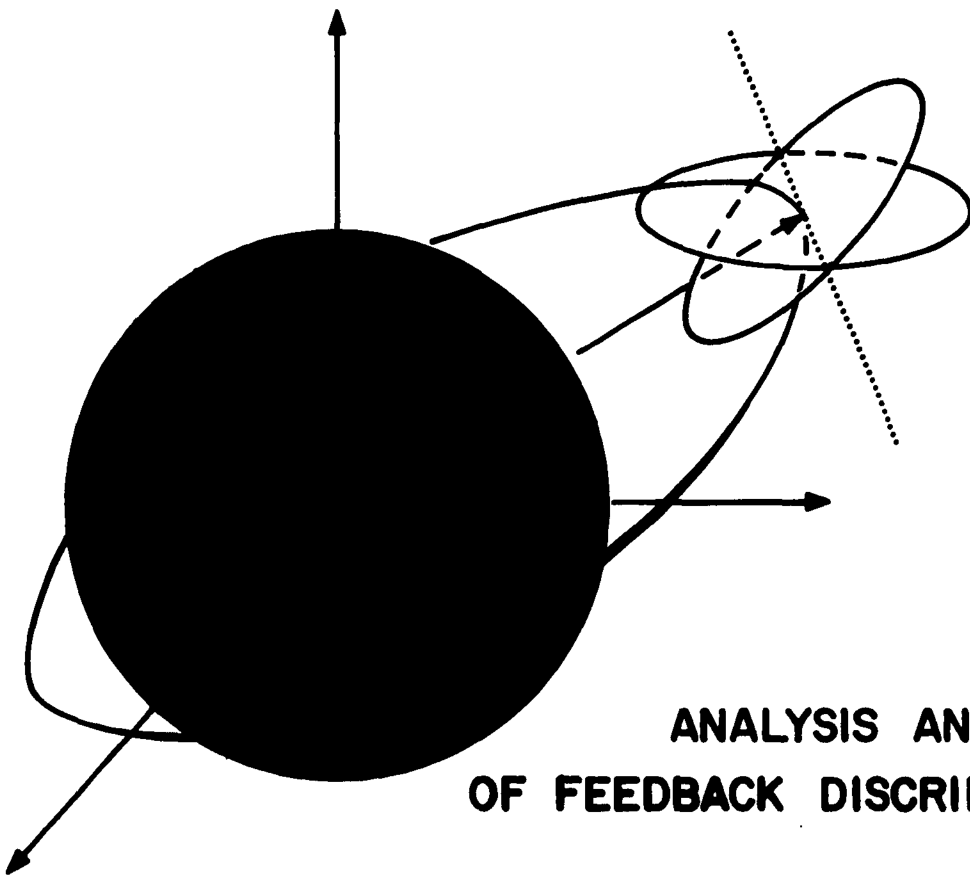

ANALYSIS AND DESIGN OF FEEDBACK DISCRIMINATORS

W. R. WOOD
ANTENNA SYSTEMS LABORATORY

AF04(695)-113

PHILCO
A SUBSIDIARY OF Ford Motor Company
WESTERN DEVELOPMENT LABORATORIES

WDL-TN62-9
31 December 1962

TECHNICAL NOTE

ANALYSIS AND DESIGN OF FEEDBACK DISCRIMINATORS

By
W. R. Wood
Antenna Systems Laboratory

Submitted by

PHILCO CORPORATION
Western Development Laboratories
Palo Alto, California

Definitive Contract AF04(695)-113

Prepared for

AIR FORCE SPACE SYSTEMS DIVISION
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Inglewood, Californa

ABSTRACT

PHILCO WDL-TN62-9 UNCLASSIFIED
ANALYSIS AND DESIGN OF FEEDBACK DISCRIMINATORS
W. R. Wood/Antenna Systems Laboratory 91 Pages
31 December 1962 Contract No. AF04(695)-113

The Analysis and Design of Feedback Discriminators applies to both systems engineering and hardware design. The report develops a linear equivalent model, analyzes the model, examines transient response, gives thresholds, presents a design procedure, and states hardware considerations. The design procedure is illustrated by an example.

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FOREWORD

This Technical Note on Definitive Contract AF04(695)-113 has been prepared in accordance with Exhibit "A" of that contract and Paragraph 4.2.2 of AFBM 58-1, "Contractor Reports Exhibit," dated 1 October 1959, as revised and amended.

This Technical Note attempts to provide the reader with an understanding of the mechanism by which feedback discriminators operate and to systematize the procedure for designing a feedback discriminator.

This document was prepared by Philco Western Development Laboratories as a supplement to the basic "Multipurpose Receiver Study," TR1850, and is fulfilling the requirements of Paragraph 1.2.1.2 of Exhibit 61-27A, "Satellite Control Subsystem Work Statement", dated 15 February 1962, as revised and amended.

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Analysis and Design of Feedback Discriminators is a Technical Note compiled in the Antenna Systems Laboratory, Surface Receiver Section, J. Del Guercio, Project Manager; the report was written by Warren R. Wood.

Recognition is given to John M. Stephenson who provided valuable technical assistance in preparation of the Technical Note and to Robert Nolte who performed much of the experimental work.

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TABLE OF CONTENTS

Abstract
Foreword
Acknowledgment
Table of Contents
List of Illustrations
List of Tables
Definition of Terms

Section		Page
1	INTRODUCTION	1-1
2	A QUALITATIVE DESCRIPTION OF FEEDBACK DISCRIMINATORS	2-1
3	LINEAR EQUIVALENT MODEL	3-1
4	TRANSFER FUNCTIONS	4-1
5	LOOP TRANSIENT RESPONSE	5-1
6	SIGNAL-TO-NOISE PERFORMANCE	6-1
7	OPTIMIZATION OF LOOP PARAMETERS	7-1
8	DESIGN PROCEDURE AND CONSIDERATIONS	8-1
9	REFERENCES	9-1
10	BIBLIOGRAPHY	10-1

APPENDICES

A	DERIVATION OF A LINEAR MODEL	A-1
B	TRANSFER FUNCTIONS	B-1
C	TRANSIENT RESPONSE	C-1
D	DERIVATION OF CLOSED LOOP NOISE BANDWIDTH	D-1
E	THRESHOLDS	E-1
F	FEEDBACK DISCRIMINATOR DESIGN	F-1
G	SIGNAL-TO-NOISE RATIO PROPERTIES OF STANDARD AND FEEDBACK DISCRIMINATORS	G-1
H	EXPERIMENTAL RESULTS	H-1

LIST OF ILLUSTRATIONS

Figure	Title	Page
2-1	Block Diagram of a Feedback Discriminator	2-2
3-1	Block Diagram of a Feedback Discriminator Showing the Input Voltage and the VCO Output Voltage	3-4
3-2	"Linearized" Equivalent Model of a Feedback Discriminator	3-5
4-1	Block Diagram Illustrating Points in the Loop for which Transfer Functions have been Determined	4-2
8-1	Schematic of a Broadband Discriminator	8-5
8-2	Block Diagram of a Loop with Double Conversion and AGC	8-6
8-3	Diagram of a Combination IF Amplifier, IF Filter (Single-Pole, Wide Band), and a Limiter	8-7
A-1	Characteristic of an Ideal Discriminator	A-5
A-2	Block Diagram Showing Bandpass Filter, Reference Oscillator, and Mixer	A-5
A-3	Linearized Loop	A-8
B-1	Diagram Showing H(s) as the Loop Transfer Function at the Summing Junction	B-6
C-1	Loop Form used in the Error Transient Analysis	C-1
D-1	Circuit whose Closed Loop Noise Bandwidth will be Determined	D-2
G-1	Required Bandwidth Divided by f_m vs β	G-7
H-1	Closed Loop Output Amplitude vs. Angular Frequency . .	H-2

LIST OF TABLES

Table	Title	Page
4-1	Loop Transfer Functions	4-3
7-1	Optimum Values of N and F for Sinusoidal Modulation	7-4
B-1	Butterworth Optimizing Form	B-2
B-2	ITAE Optimizing Form	B-3
E-1	Values of Discriminator Input SNR at Threshold for Various Ratios of B_{NIF} to f_m	E-5

DEFINITION OF TERMS

Symbol	Terminology	Unit
A_c	Maximum carrier amplitude	volts
A_n	Maximum incremental noise amplitude	volts
a	Equivalent IF pole	rad/sec
B_n	Loop natural resonant frequency	cps
B_{NIF}	IF noise bandwidth (low pass equivalent)	cps
B_{nl}	Loop noise bandwidth	cps
B_{NRF}	RF noise bandwidth (low pass equivalent)	cps
b	Low pass filter pole	rad/sec
$C_{IF}(t)$	Time varying IF signal	volts
C(s)	Loop S-plane output (at low pass filter output)	volts/rad/sec
$C_s(t)$	Time varying input signal	volts
c(t)	Output time response	volts
E(s)	Loop S-plane error	cps/rad/sec
E_2	Maximum VCO amplitude	volts
F	Feedback gain factor	unity
$F_1(s)$	Equivalent S-domain loop input	cps/rad/sec
F(s)	Arbitrary filter input	volts/rad/sec
f_d	Carrier frequency deviation	cps
$f_{in}(t)$	Total loop time domain loop input (frequency)	cps
$f_1(t)$	Equivalent time domain loop input	cps
f_m	Modulation frequency	cps
f(t)	Inverse Laplace transform of F(s)	volts

DEFINITION OF TERMS (cont'd)

Symbol	Terminology	Unit
$G(s)$	Arbitrary filter transfer function	unity
$G_a(s)$	Equivalent IF low pass transfer function	unity
$G_b(s)$	Low pass filter and gain constant transfer function	unity
$G_d(s)$	Desired transfer function	volts/rad/sec
$G_f(f)$	Equivalent noise spectral density	cps
$G_o(s)$	Weiner optimum transfer function	volts/rad/sec
$H(jw)$	$H(s)$ with $s = jw$	cps/rad/sec
H_m	Maximum absolute value of VCO output transfer function	unity
$H(s)$	S-plane VCO output	cps/rad/sec
$J_n(\beta)$	Bessel function of first kind n^{th} order and argument β	unity
K_A	Amplifier gain constant	unity
K_d	Discriminator transfer constant	volts/cps
K_v	VCO transfer constant	cps/volt
$K_{1,2,etc.}$	Arbitrary residues	unity
N	Ratio of equivalent IF pole to low pass pole	unity
N_o	Noise power spectral density normalized in terms of frequency (two-sided)	cps
N_o'	Noise power spectral density (two-sided)	watts/cps
N_o''	Normalized noise power spectral density (two-sided)	rad^2 cps
$n(t)$	Time varying incremental noise	volts

DEFINITION OF TERMS (cont'd)

Symbol	Terminology	Unit
P_N	Noise power	watts
P_s	Signal power	watts
$R(s)$	S-plane loop input	cps/rad/sec
$r(t)$	Input time function	cps
SNR	Signal-to-noise ratio	unity
t	Time	sec
$W(s)$	Discriminator output in S-plane	volts/rad/sec
$w(t)$	Time domain discriminator output	volts
β	Modulation index	unity
β_{IF}	IF modulation index	unity
β_{RF}	Input modulation index	unity
$\epsilon(t)$	Time domain loop error	cps
ζ	Loop damping factor	unity
θ_{in}	Loop input phase	rad
θ_2	Extraneous VCO phase	rad
$\mu_2(t)$	Time weighting function	unity
ξ	Threshold improvement factor	db
ρ_{DT}	Discriminator threshold SNR	unity
ρ_i	Loop input SNR	unity
ρ_{iD}	Discriminator input SNR	unity
ρ_{iT}	Loop input threshold SNR	unity
ρ_o	Output SNR	unity

DEFINITION OF TERMS (cont'd)

Symbol	Terminology	Unit
$\hat{\phi}$	VCO signal phase estimate	rad
ϕ_n	Input noise phase	rad
ϕ_1	Input information phase	rad
ω_c	Carrier angular frequency	rad/sec
ω_i	Discriminator input angular frequency	rad/sec
ω_m	Angular modulation frequency	rad/sec
ω_n	Angular noise frequency	rad/sec
ω_{nl}	Loop noise bandwidth (angular frequency)	rad/sec
ω_o	IF center angular frequency	rad/sec
ω_x	Angular frequency step input	rad/sec
$\mathcal{F}$	Fourier transform	
$\mathcal{F}^{-1}$	Inverse Fourier transform	
$\mathcal{L}$	Laplace transform	
$\mathcal{L}^{-1}$	Inverse Laplace transform	

SECTION 1
INTRODUCTION

1.1 GENERAL

This report on feedback discriminators was undertaken in order to present a systematic method of analyzing and designing feed back discriminators. The report is hopefully written in a manner whick can be readily understood and applied to problems encountered in designing communication systems.

Historically speaking, feedback discriminators date back to the early days of FM. For example, Chaffee and Carson, two of the early pioneers in FM feedback, published papers dealing with feedback discriminators in The Bell System Technical Journal as early as 1939. But until recently, feedback discriminators have been little more than laboratory curiosities. However, as of late, feedback discriminators have been successfully employed in projects Echo and Telstar.

While the feedback discriminator performs the same function as a standard discriminator in that both devices demodulate frequency-modulated signals, the feedback discriminator has a lower closed loop threshold than the standard discriminator. This threshold lowering is the most important advantage obtainable through using a feedback discriminator.

A designer can use this threshold lowering to his advantage in several ways. For example, he can reduce the amount of transmitted power and thus realize a monetary saving. On the other hand, he can transmit the same power, increasing transmission reliability. He can also increase the modulation index while keeping the transmitted power constant, and the resulting spreading of the RF spectrum increases detection security.

SECTION 2
A QUALITATIVE DESCRIPTION OF FEEDBACK DISCRIMINATORS

2.1 GENERAL

Before proceeding with a detailed discussion of feedback discriminators, it seems advisable to discuss qualitatively their operation.

In carrying on this discussion, a block diagram of a feedback discriminator will be utilized, Figure 2-1. The feedback discriminator employs a mixer, an IF amplifier with a bandpass filter, a frequency detector (discriminator), a low pass filter, and a VCO (voltage-controlled oscillator). An FM signal is applied to the mixer at the input terminal, and the demodulated base band is obtained at either point W or point C.

An input signal is multiplied by an output signal from the VCO in the mixer, and the resultant signal, which is an FM signal, is applied to the IF amplifier. The IF center frequency is equal to the difference between the input carrier frequency and the VCO center frequency.

The purpose of the amplifier is to provide sufficient voltage to the discriminator to enable it to operate in a linear manner. The bandpass filter attenuates noise outside the bandwidth of the IF signal. A standard discriminator is then used to demodulate the FM signal, and the resulting demodulated baseband is passed through a low pass filter to achieve further noise attenuation.

The output of the low pass filter is applied to the VCO. Since voltage changes at the VCO input manifest themselves as frequency changes at the VCO output, the VCO is essentially a frequency modulator. The modulation index of the VCO output is a function of the loop gain, while the baseband is an estimate of the input signal. Moreover, two FM signals are then beaten together in the mixer, and since the phase components subtract in the mixing process, the equivalent modulation

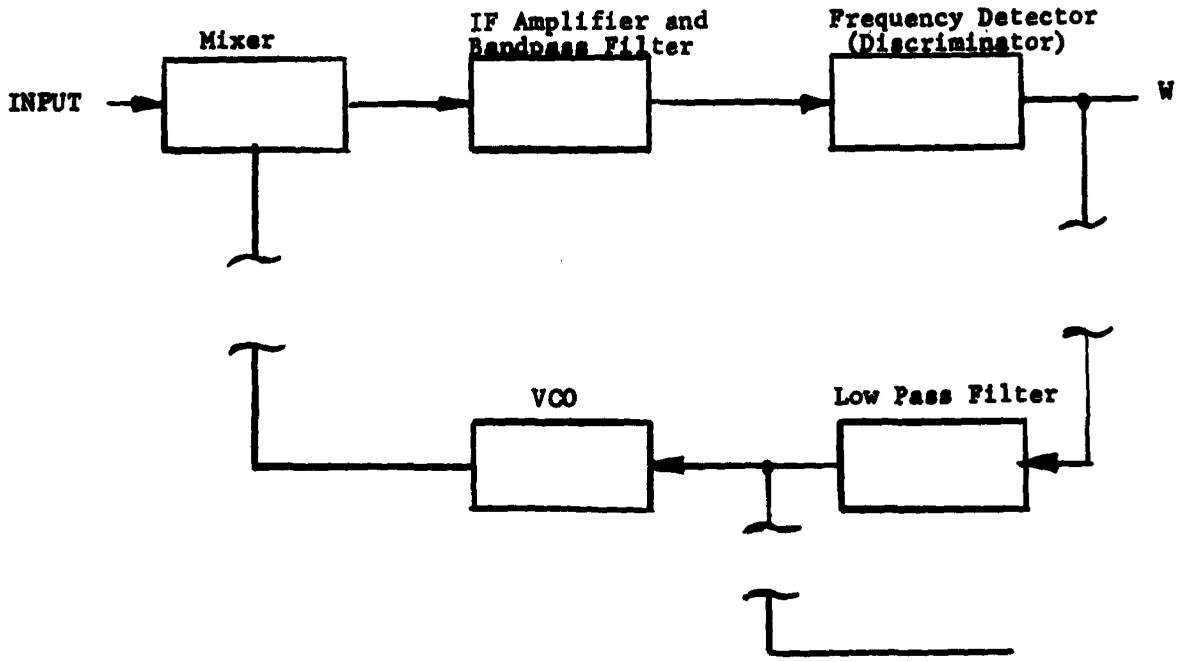

Fig. 2-1. Block Diagram of a Feedback Discriminator

index of the signal at the output of the mixer has been reduced.

With this modulation index reduction, fewer baseband sidebands are needed to convey the information. Consequently, the IF bandpass filter can be narrower than if no feedback were employed. This reduction in noise bandwidth seen by the discriminator accounts for the threshold reduction achievable with a feedback discriminator.

In the analysis of feedback discriminators, much of the effort has been directed to development of the idealized situation in which the loop has only two poles (one pole is contributed by the IF filter and the other pole is produced by the low pass filter). Although a two-pole loop constitutes an optimum design from the standpoint of time-response and stability, a physically realizable design will have more than two poles; however, careful attention to design details will render the effects of the additional poles negligible.

SECTION 3
LINEAR EQUIVALENT MODEL

3.1 GENERAL

The first step in analyzing feedback discriminators is the reduction of the loop parameters to their "linear equivalent" forms. This "linearization" process can be circumvented in the analysis of feedback discriminators (for example, the analytical development can be carried out using differential equations), but "linearization" greatly simplifies the work entailed in analyzing and designing feedback discriminators.

The "linearization" process is here defined as being the reduction of loop parameters to S-plane or Laplace polynomials when the input to the loop is considered as being frequency. A detailed development of the "linearization" is contained in Appendix A.

The loop components prior to "linearization" are shown in Figure 3-1. At the VCO output, E_2 is the maximum VCO voltage amplitude; $\omega_2/2\pi$ is the VCO center frequency; $\hat{\phi}$ is the VCO estimate of the input signal phase; θ_2 is a phase term encompassing noise jitter as well as all VCO phase components not contained in $\omega_2 t$ and $\hat{\phi}$. At the input to the loop, A_c is the maximum voltage amplitude of the signal; $\omega_c/2\pi$ is the carrier frequency; ϕ_1 is the phase of the information; ϕ_n is the noise phase contribution. The equivalent frequency input is found through differentiating the input phase:

$$F_{in} = \frac{\omega_c + \frac{d\phi_1}{dt} + \frac{d\phi_n}{dt}}{2\pi} \tag{3-1}$$

When sinusoidal modulation is used, the Laplace transform of equation (3-1) is

$$F_1(s) = \frac{\omega_c}{2\pi s} + \frac{f_d\, \omega_m}{s^2 + \omega_m^2} + \frac{1}{2\pi} \quad \frac{A_n}{A_c} \, \frac{\omega^2}{s^2 + \omega^2} \qquad (3\text{-}2)$$

where

f_d = carrier frequency deviation

$\omega_m/2\pi$ = modulation frequency

A_n = Maximum incremental noise voltage amplitude

The mixer shown in figure 3-1 multiplies input voltage by VCO output voltage. Therefore, input and VCO frequency components subtract and add, but the resulting addition component is neglected since a bandpass filter centered at $\omega_o = \omega_c - \omega_2$ follows the mixer. The subtraction component is known as frequency error.

Before discussing "linearization" of the bandpass filter, the discriminator will be considered. While a discriminator generally has an "S"-curve response (i.e., output voltage amplitude vs. input frequency), generally, the transfer characteristic is approximately linear throughout an operating range about the center frequency. $\omega_o/2\pi$, the IF center frequency, is selected as the discriminator center frequency; and at $\omega_o/2\pi$, the distriminator output voltage is zero, but the discriminator output voltage is K_d volts when

$$\frac{\omega_i - \omega_o}{2\pi} = 1$$

Hence, the discriminator transfer function is K_d in the linear range.

As far as the band filter is concerned, it is converted to an equivalent low pass function, defined as $G_a(s)$, through a transformation which is detailed in Appendix A. In this transformation, the IF or bandpass filter center frequency is converted from $\frac{\omega_o}{2\pi}$ to zero.

For example, if the IF bandwidth is 2a rad/sec and if the filter is a single-pole filter;

$$G_a(s) = \frac{a}{s + a} \tag{3-3}$$

Notice that the bandwidth is reduced a factor of two by the transformation.

A low pass filter and a video amplifier follow the discriminator in the loop. The function of $G_b(s)$ is defined so that it encompasses these two quantities. If the low pass filter is a simple 6 db/octave roll-off filter and if the video amplifier has a voltage gain of K_A, then

$$G_b(s) = \frac{K_A b}{S + b} \tag{3-4}$$

A VCO changes its output frequency of oscillation as a function of its input voltage. In the feedback discriminator circuit, it is assumed that the oscillatory frequency is directly proportional to the transfer function of the VCO which is K_V.

Now that the loop components have been "linearized," a composite model can be constructed, and this model is illustrated in Figure 3-2. For this model, $G_a(s)$ and $G_b(s)$ assume the values denoted by equations (3-3) and (3-4), respectively. The model shown in Figure 3-2 will be used as a basis for much of the analysis in following sections.

Two points worth stressing are: (1) All noise bandwidths are double-sided since a two-sided noise spectrum is used. (2) The input to the loop goes through a limiting process before reaching the loop. Limiting eliminates amplitude variations.

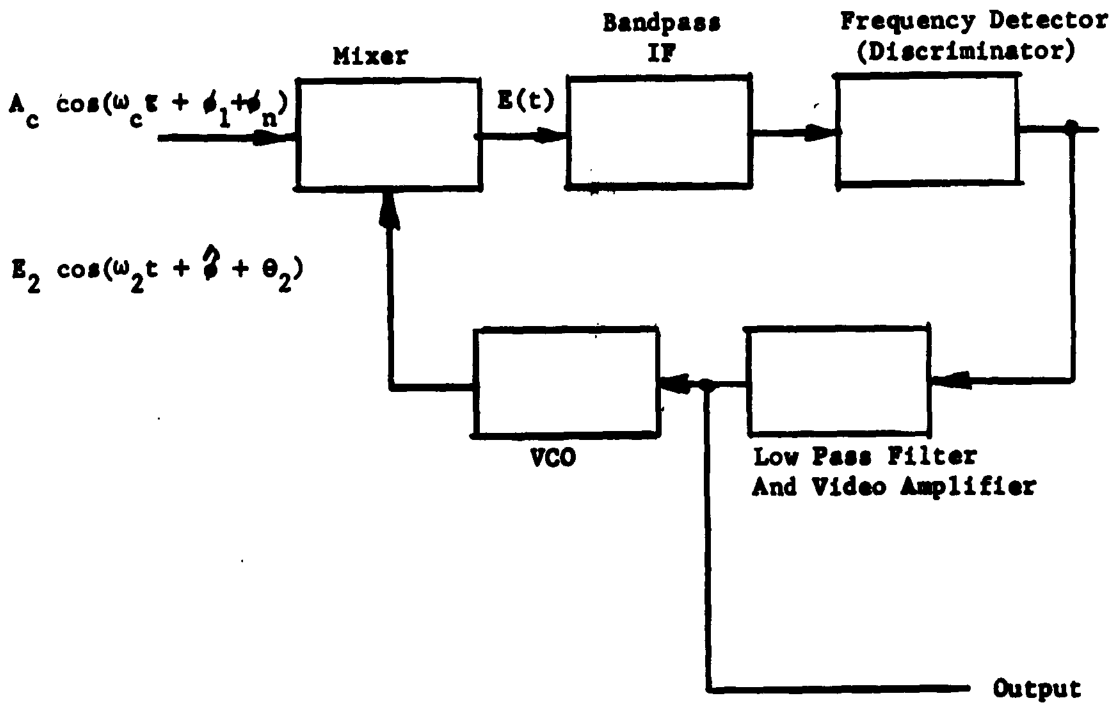

Fig. 3-1 Block Diagram of a Feedback Discriminator Showing the Input Voltage and the VCO Output Voltage

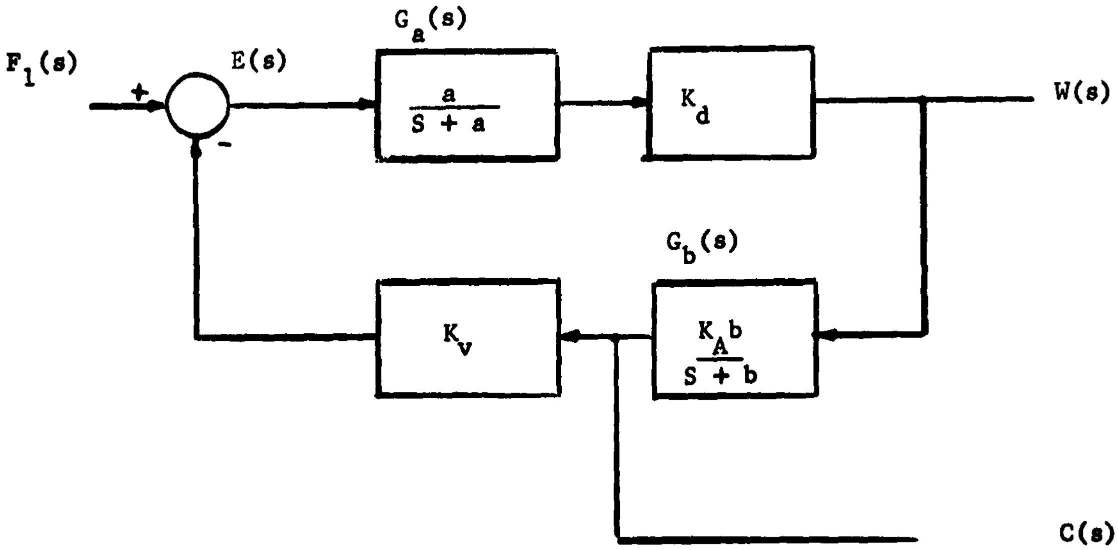

Fig. 3-2 "Linearized" Equivalent Model of a Feedback Discriminator

SECTION 4
TRANSFER FUNCTIONS

4.1 GENERAL

In view of the analytical necessity of being able to express loop response in terms of input stimuli, this section is devoted to presenting loop transfer functions in terms of an equivalent frequency input. Appendix B contains details associated with deriving these transfer functions.

Referring to Figure 4-1, loop transfer functions for $\frac{E(s)}{R(s)}$, $\frac{W(s)}{R(s)}$, $\frac{C(s)}{R(s)}$, $\frac{H(s)}{R(s)}$ have been found. R(s) denotes the Laplacian frequency input, E(s) is the mixer output or error, W(s) is the discriminator output, C(s) is the low pass filter output, and H(s) is the VCO output. Table 4-1 gives these transfer functions. Here the general case with the equivalent IF low pass transfer function of $G_a(s)$ and the low pass filter and gain constant transfer function of $G_b(s)$ is presented as well as the two-pole case in which $G_a(s) = \frac{a}{S + a}$ and $G_b(s) = \frac{K_A b}{S + b}$.

In Table 4-1, the following substitutions are used:

$$\left.\begin{aligned} \omega_n^2 &\triangleq ab\,(1 + K_v K_d K_A) \\ 2\zeta\omega_n &\triangleq a + b \end{aligned}\right\} \qquad (4\text{-}1)$$

Table 4-1 can be utilized to find the transfer function of any loop position except the output of the IF filter. If in the loop being analyzed; $G_a(s) \neq \frac{a}{S + b}$ and/or $G_b(s) \neq \frac{K_A b}{S + b}$, then the transfer characteristics are determined by substituting the correct $G_a(s)$ and $G_b(s)$ factors into the second column of Table 4-1.

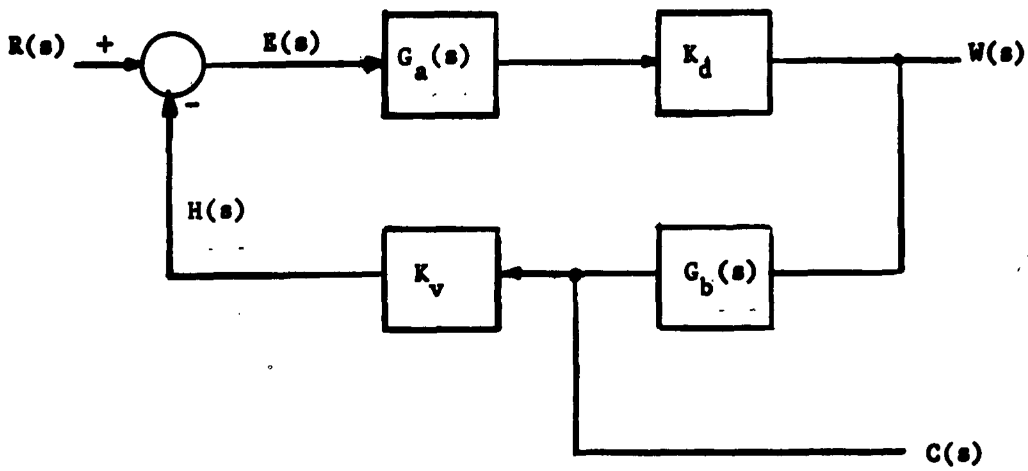

Fig. 4-1 Block Diagram Illustrating Points in the Loop for which Transfer Functions have been Determined

TABLE 4-1

LOOP TRANSFER FUNCTIONS

LOOP TRANSFER QUANTITY	TRANSFER FUNCTION FOR GENERAL CASE	TRANSFER FUNCTION FOR TWO-POLE CASE
$\frac{E(S)}{R(S)}$	$\frac{1}{1 + G_a(S)G_b(S)K_dK_v}$	$\frac{S^2 + 2\zeta\omega_n S + ab}{S^2 + 2\zeta\omega_n S + \omega_n^2}$
$\frac{W(S)}{R(S)}$	$\frac{G_a(S)K_d}{1 + G_a(S)G_b(S)K_dK'_v}$	$\frac{(S + b)\,aK_d}{S^2 + 2\zeta\omega_n S + \omega_n^2}$
$\frac{C(S)}{R(S)}$	$\frac{G_a(S)G_b(S)K_d}{1 + G_a(S)G_b(S)K_dK_v}$	$\frac{ab\ K_dK_A}{S^2 + 2\zeta\omega_n S + \omega_n^2}$
$\frac{H(S)}{R(S)}$	$\frac{G_a(S)G_b(S)K_dK_v}{1 + G_a(S)G_b(S)K_dK_v}$	$\frac{\omega_n^2 - ab}{S^2 + 2\zeta\omega_n S + \omega_n^2}$

SECTION 5
LOOP TRANSIENT RESPONSE

5.1 GENERAL

Transient loop error for sine wave modulation and a frequency step will be presented in this section (detailed derivation is contained in Appendix C). Transient error produced by input frequency stimuli is of interest for several reasons.

One reason the transient error merits consideration is that the transient output of the loop can be readily determined from the transient error. The equation expressing this relationship is:

$$c(t) = \frac{1}{K_v} \left[r(t) - \epsilon(t) \right] \qquad (5\text{-}1)$$

where,

$c(t)$ = output time response in volts

$\epsilon(t)$ = error time response in cps

$r(t)$ = input time function in cps

Another argument for considering transient error is the effect transient error has on the ability of the loop to follow input frequency swings. For example, if the error frequency is sufficiently offset from the IF center frequency, then the loop gain will be less than one, and the loop will not remain locked to the input.

Assuming a sinusoidal input whose voltage form is:

$$\rho_S(t) = A_c \sin \left(\omega_c t + \frac{f_d}{f_m} \cos \omega_m t\right) \qquad (5\text{-}2)$$

where,

f_d = carrier frequency deviation

The equivalent frequency input of equation (5-2) is then:

$$f_1(t) = f_d \sin \omega_m t \tag{5-3}$$

With the equivalent frequency input given in equation (5-3), the traisient error is:

$$\epsilon(t) = f_d \, \omega_m \left[\frac{1}{\omega_m} \sqrt{\frac{4\zeta^2 \omega_n^2 \omega_m^2 + \left(ab - \omega_m^2\right)^2}{4\zeta^2 \omega_n^2 \omega_m^2 + \left(\omega_n^2 - \omega_m^2\right)^2}} \sin(\omega_m t + \psi_1) \right]$$

$$+ \frac{e^{-\zeta \omega_n t}}{\omega_n \sqrt{1-\zeta^2}} \sqrt{\frac{\left(ab - \omega_r^2\right)^2}{4\zeta^2 \omega_n^2 \omega_m^2 + \left(\omega_n^2 - \omega_m^2\right)^2}}$$

$$\sin \left(\omega_n \sqrt{1 - \zeta^2} \, t + \psi_2\right) \text{ in cps} \tag{5-4}$$

where,

$$\psi_1 = \tan^{-1} \left(\frac{2\zeta \omega_n \omega_m}{ab - \omega_m^2} \right) - \tan^{-1} \left(\frac{2\zeta \omega_n \omega_m}{\omega_n^2 - \omega_m^2} \right)$$

$$\psi_2 = \tan^{-1} \left(\frac{2\zeta \omega_n^2 \sqrt{1 - \zeta^2}}{\omega_n^2 (2\zeta^2 - 1) + \omega_m^2} \right)$$

Another input stimulus that will be treated is a frequency step. Such a phenomenon is characteristic of FSK transmission. However, the steady state value of this transient error will be equal to the steady state error produced by any carrier shift, regardless of how the shift takes place. With a frequency step, the input signal voltage is:

$$e_S(t) = A_c \sin \left[\omega_c t + \mu_2(t)\, \omega_x t\right] \qquad (5\text{-}5)$$

where;

$\mu_2(t) = 1$ for $t \geq 0$

$\mu_2(t) = 0$ for $t < 0$

In terms of an equivalent frequency input, equation (5-5) reduces to:

$f_1(t) \frac{\omega_x}{2\pi}$ for $t \geq 0$.

The Laplacian input is:

$$F_1(s) = \frac{\omega_x}{2\pi S} \qquad (5\text{-}6)$$

When the input stimuli has the form displayed in equation (5-6), the transient error is as follows:

$$\epsilon(t) = \frac{\omega_x}{2\pi} \left[\frac{1}{F} - \frac{(1 - F)}{F\sqrt{1-\zeta^2}}\, e - \zeta\, \omega_n t \sin(\omega_n \sqrt{1 - \zeta^2} t + \psi)\right] \text{in cps.} \qquad (5\text{-}7)$$

where;

$$\psi = \tan^{-1} \sqrt{\frac{1-\zeta^2}{\zeta^2}}$$

Explicit transient error solutions have been given for a sinusoidal input and for a step input, but if a solution for transient error resulting from another form of input is desired, the Laplacian technique illustrated in Appendix C can generally be applied to the problem.

SECTION 6
SIGNAL-TO-NOISE PERFORMANCE

6.1 GENERAL

This section has as its primary objective the discussion of the threshold improvement achievable through using a feedback discriminator. However, a comparison will be made of the output signal-to-noise ratio (above threshold) of a standard discriminator and a feedback discriminator. Appendix E contains the detailed derivations converning the threshold problem while Appendix G is devoted to the details concerning output signal-to-noise ratios (SNR).

A feedback discriminator has essentially two thresholds. One is the open loop threshold, which is the threshold of the standard discriminator in the loop; the other is the closed loop threshold. If the open loop threshold is reached prior to the closed loop threshold, then the threshold of the loop is determined by the IF SNR. However, if the closed loop threshold is reached first, then the threshold of the loop is determined by the closed loop threshold alone.

6.2 FORMULAS

The threshold of standard discriminator is defined as the lowest value of input SNR, with fixed IF noise bandwidth and modulation frequency, for which the output SNR is given by the following formula:

$$\rho_o = \frac{3}{2}\,\beta^2_{RF}\,\frac{B_{NIF}}{f_m}\,\rho_i \qquad (6\text{-}1)$$

where;

B_{NIF} = IF noise bandwidth (low pass equivalent)

f_m = Modulation frequency

ρ_o = Output SNR

β_{RF} = Modulation index

Table E-1 of Appendix E lists discriminator thresholds for various values of $B_{NIF}/H\ f_m$.

Moreover, if the discriminator in the loop reaches its threshold prior to the point at which the closed loop threshold is reached, then the threshold of the loop is:

$$\rho_{iT} = \rho_{DT} \frac{B_{NIF}}{B_{nl}}$$

where;

ρ_{iT} = Input SNR at threshold

ρ_{DT} = Discriminator SNR threshold

B_{nl} = Closed loop noise bandwidth

Since $B_{NIF} < B_{nl}$, it is obvious from equation (6-2) that some degree of threshold improvement has been realized. However, a greater threshold improvement is generally obtained by reaching the closed loop threshold first. The closed loop threshold is given by the following equation which is defined in noise bandwidth B_{nl}:

$$\rho_{iT} \approx 4.8\left(\frac{F-1}{F}\right)^2 \qquad (6\text{-}3)$$

where;

$F = 1 + K_v\ K_d\ K_A$

To ensure that the open loop threshold is not reached prior to the point at which the closed loop threshold is reached, it is necessary for the SNR at the discriminator input to be aboveits threshold when the loop input SNR is equal to the closed loop threshold SNR. At the closed loop threshold, the discriminator input SNR is:

$$\rho_{iD} = \rho_{iT} \frac{B_{nl}}{B_{NIF}} \tag{6-4}$$

where:

ρ_{iD} = discriminator input SNR.

Substituting the right side of equation (6-3) for ρ_{iT} into equation (6-4) yields:

$$\rho_{iD} \approx 4.8 \left(\frac{F-1}{F}\right)^2 \frac{B_{nl}}{B_{NIF}} \tag{6-5}$$

Comparing equation (6-5) with Table E-1, it is seen that the open loop threshold is satisfied if B_{nl} = 2.18 B_{NIF} and F = 6.

For example, suppose a carrier is being modulated by a 1 Mc wave with a modulation index of 10. Then the ratio of B_{NRF}/f_m is equal to 24, and the threshold of an ordinary discriminator is about 13 db (see Table E-1). But if a feedback discriminator with F = 10 is used, the threshold is calculated from equation (6-3) to be 3.92 or 5.94 db. Moreover, since B_{nl} will be smaller than the 24 Mc input noise bandwidth, a further threshold enhancement results. Equation (6-6) expresses total threshold improvement as follows:

$$\zeta = 13 - 5.94 + 10 \log_{10} \frac{24 \text{ Mc}}{B_{nl}} \tag{6-6}$$

where:

ζ = threshold improvement factor in db

From Appendix G, the equation for SNR at the output of a feedback discriminator with post-detection filtering is:

$$\rho_o = \frac{12\pi^3 \, (\beta_{RF} \, f_m)^2 \, \omega_n^4}{\omega_m^3 \left[4\zeta^2 \, \omega_n^2 \, \omega_m^2 + (\omega_n^2 - \omega_m^2)^2 \right]} \qquad B_{NRF}, \; \rho_i$$

But assuming $\omega_n >> \omega_m$ and $b = 2\pi f_m$, the output SNR becomes:

$$\rho_o \approx \frac{3 \; \beta_{RF}^2}{2} \left(\frac{B_{NRF}}{f_m} \right) \rho_i \qquad (6\text{-}7)$$

To summarize the results implicit in Section 6, a feedback discriminator has a lower threshold than a standard discriminator. But the SNR of a standard discriminator is proportional to β_{RF}^2 while the SNR of a feedback discriminator is also a function of β_{RF}^2. (ρ_i is defined in B_{NRF}).

SECTION 7
OPTIMIZATION OF LOOP PARAMETERS

7.1 GENERAL

In designing a feedback discriminator or in examining an existing design, it is useful to know what the optimized loop parameters are. Knowing the optimum parameters, a designer attempts to make his design approximate the optimum configuration. Moreover, the "goodness" of an existing design can be ascertained by comparing it with the optimum design.

An optimum loop has only two poles; the reason for this is twofold. Additional poles reduce the phase margin and deleteriously affect time response (see Appendix B). If the phase margin is reduced by additional poles, there is the possibility that the loop will oscillate. On the other hand, additional poles will slow down the time-response of the loop.

As far as loop parameters are concerned, there are three parameters to optimize. Loop damping, loop gain, and filter bandwidth ratio are the three parameters. Taking damping first, there are several methods of arriving at an optimum damping factor.

Unfortunately, each method leads to a different answer when more than two poles are involved. Two optimizing techniques, the Butterworth and ITAE forms are discussed in Appendix B.

Table B-1 in Appendix B gives Butterworth optimizing forms for loops having from one to five poles while Table B-2 contains ITAE optimizing forms for loops with one to five poles. When a loop has two poles, the two forms agree that the optimum loop transfer function denominator is as follows:

$$D(s) = S^2 + 1.4\,\omega_n S + \omega_n^2 \qquad (7\text{-}1)$$

Equation (7-1) means that $2\zeta = 1.4$, or $\zeta \approx 0.707$ where ζ is the loop damping factor. It is not always possible to have a loop damping of 0.707 due to limitations imposed by the open and closed loop thresholds as well as closed loop noise bandwidth. For example, the requirements imposed by the open loop threshold that $B_{nl} \gg 2.5\ B_{NIF}$ imposes the following restriction on ζ (see Section 6):

$$\zeta \leq \frac{1}{5} \sqrt{\frac{F}{N}} \tag{7-2}$$

where $N = \frac{a}{b}$

$$F = 1 + K_v\ K_d\ K_A$$

Equation (7-2) imposes an upper boundary on ζ. In general, the relation between ζ, N, and F is:

$$\zeta = \frac{N + 1}{2\sqrt{FN}} \tag{7-3}$$

As far as F is concerned, an F high enough to give sufficient bandwidth compression should be used. However, F should be made as small as possible since an increase in F will raise the closed loop threshold [see equation (6-3)] and will increase the closed loop noise bandwidth as shown in the following equation:

$$B_{nl} = \frac{b \qquad NF}{2\ (N + 1)} \tag{7-4}$$

where:

B_{nl} = closed loop noise bandwidth.

The IF noise bandwidth must be narrow enough to satisfy the open loop threshold. However, it cannot be a sharp cutoff filter due to stability considerations (see Appendix B), and it must be sufficiently wide to keep signal distortion at an acceptable level. Feedback tends to

reduce distortion; hence it is possible to let $N = \beta_{IF}$ for the entire range of β_{IF} instead of the value indicated by Figure G-1.

Loop stability is increased as β_{IF} is increased. Additionally, the IF filter bandwidth should be made as wide as possible, considering the open loop threshold, so that ζ can be made to approach 0.707 (see equation 7-3).

Balancing the preceding considerations, Table 7-1 gives optimum values of F and N for values of β_{RF} starting at $\beta_{RF} = 3$; if the expected β_{RF} is less than 3, an ordinary discriminator should be used. Here F is not allowed to exceed 50 because a value greater than 50 could lead to loop instability, and a minimum practical discriminator threshold of 9 db is assumed.

TABLE 7 - 1

OPTIMUM VALUES OF N and F for
SINUSOIDAL MODULATION

β_{RF}	$N = \beta_{IF}$	F
$3 < \beta_{RF} \leq 10$	1	$F = 2\beta_{RF}$
$10 < \beta_{RF} \leq 29$	2	$F = \beta_{RF}/1.26$
$29 < \beta_{RF} \leq 46$	3	$F = \beta_{RF}/2$
$46 < \beta_{RF} \leq 80$	4	$F = \beta_{RF}/2.7$
$80 < \beta_{RF} \leq 118$	5	$F = \beta_{RF}/3.47$
$118 < \beta_{RF} \leq 167$	6	$F = \beta_{RF}/4.2$
$167 \quad \beta_{RF} \leq 223$	7	$F = \beta_{RF}/4.9$
$223 < \beta_{RF} \leq 280$	8	$F = \beta_{RF}/5.6$
$280 < \beta_{RF}$	$\frac{\beta_{RF}\sqrt{2}}{50}$	$F = 50$

SECTION 8
DESIGN PROCEDURE AND CONSIDERATIONS

8.1 DESIGN PROCEDURE

Designing is made a tricky proposition by the open loop threshold and by the need to maintain an adequate phase margin throughout the operating range. Moreover, optimizing loop parameters is a design goal. Toward this end, optimization as discussed in Section 7 is applied to the design method.

Three quantities must be known before a design can be undertaken; they are: the modulation frequency, expected carrier frequency, and input β. Knowing these three quantities, the design steps proceed as follows:

a. F is obtained from Table 7-1. Since $F = 1 + K_v K_d K_A$, the product of the discriminator constant, the VCO constant, and the dc amplifier gain are known.

b. N is also obtained from Table 7-1.

c. The IF center frequency, f_o, is selected so that it is at least 100 times higher than the modulation frequency (this high ratio ensures that IF discriminator leakage is sufficiently attenuated by the low pass filter to prevent its perturbing the loop VCO).

d. A single-pole filter is used as the loop IF filter, the 3 db bandwidth of this filter is $2Nf_m$ (the filter center frequency coincides with the IF center frequency).

e. The discriminator is designed to be as linear as possible (at least 5% linearity is desirable throughout the operating range) and the discriminator is designed so that its bandwidth is at least 5 Nf_m in order to minimize the effects of discriminator poles. Discriminator output impedance should be resistive.

f. A single-pole (no zero) low pass filter follows the discriminator, and discriminator output resistance is essentially an integral part of the filter. The filter 3 db cutoff frequency should be equal to f_m.

g. An amplifier whose 3 db pass band goes from dc to at least 10 f_m follows the low pass filter. This amplifier must have a high input impedance to prevent its loading down the filter. Amplifier gain, K_A, is selected so that F satisfies the condition set forth in step a.

h. Following the amplifier in the loop is a VCO. This VCO has a center frequency which is f_o cps lower than the expected input carrier frequency; thus, the difference between VCO and carrier is the IF center frequency. Additionally, the VCO should have a bandwidth of at least 5 $Ff_m \beta_{RF}$ and a 10% transfer linearity for an input range of dc to f_m.

i. Design of the loop mixer is critical only insofar as the bandwidths are concerned. The mixer must be able to accommodate the signals from the input as well as from the VCO. Moreover, the mixer output bandwidth should be at least 10 Nf_m.

8.2 A DESIGN EXAMPLE

Suppose that a 100 Mc carrier is being modulated by 100 kc signal with a deviation of 2 Mc; this means β_{RF} = 20. The design now proceeds as follows:

a. F = 16 from Table 7-1, and therefore $K_V K_d K_A = 9$.

b. N = 2 also from Table 7-1.

c. An IF center frequency of 100 x 100 Kc = 10 Mc is selected.

d. The IF filter is centered at 10 Mc and has a 3 db bandwidth of 400 kc.

e. The discriminator has a 0-volt output at 10 Mc with a 5% linearity in the $\pm$ 200 kc range about 10 Mc, and the discriminator bandwidth, which is determined by the knees in the S-curve, is 5 Mc. The output impedance is resistive (no capacitors are used here).

f. The low pass filter should be 3 db down at 100 kc. Assuming the discriminator output resistance is 10 KΩ, a 160 pf capacitor is attached between the discriminator output and ground to give the required low pass function.

g. The dc amplifier is flat, within 3 db, from dc to 1 Mc. Amplifier gain is $K_A = \frac{18}{K_d\ K_v}$.

h. VCO center frequency is set at 90 Mc, and the VCO has 10% transfer function linearity between 88.2 Mc and 91.8 Mc at its output. VCO output bandwidth is at least 9 Mc.

8.3 HARDWARE CONSIDERATIONS

Building hardware—satisfying design requirements can be difficult. For example, some IF will leak through the discriminator. If the low pass filter does not sufficiently attenuate this IF leakage to prevent its perturbing the VCO, then additional circuitry is needed to attenuate the IF. This additional IF attenuation can be provided by a low pass function (of course, the cut-off frequency must be considerably higher than f_m to minimize phase shift) such as an m-derived filter or a series-resonant "trap" circuit tuned to the IF.

The discriminator itself should have a wide bandwidth so that its phase contribution is minimized. While an ordinary ratio detector normally will not provide a large bandwidth, the circuit shown in Figure 8-1

will provide a 50% bandwidth. In this circuit, it is necessary that $R_2 \gg R_1$.

Often one finds it desirable to apply AGC to the IF amplifiers preceding the loop. The method of obtaining the AGC is shown in Figure 8-2. A limiter is placed inside the loop instead of outside the loop as is customary. This enables one to take advantage of the narrowed-loop IF bandwidth in obtaining AGC.

Obviously, the limiter will produce harmonics which would deleteriously affect the discriminator and so the limiter must be filtered. In order to minimize effects of additional poles, the limiter filtering is accomplished by a low-Q filter (see Figure 8-3). The filter elements are used to tune out the diode junction capacitance, C_j. However, the following requirements must be met:

a. $C_2 \gg C_3$

b. $C_3 \gg C_j$ maximum

c. The voltage swing must be great enough for CR_1 and CR_2 to clip.

Another feature illustrated in Figure 8-2 is the use of pads to prevent interaction. For example, a pad is inserted between the loop VCO and the mixer. This isolation prevents either local oscillator frequencies or input frequencies from disturbing the VCO (of course, the use of balanced mixers will also help to alleviate the VCO interaction problem). A pad is also employed in the AGC circuit so that it does not disturb the loop.

Section 8 developed a design procedure predicated on the optimized parameters of Section 7, and a design example to elucidate the procedure was presented. Additionally, some hardware problems that would crop up in implementing a design were considered.

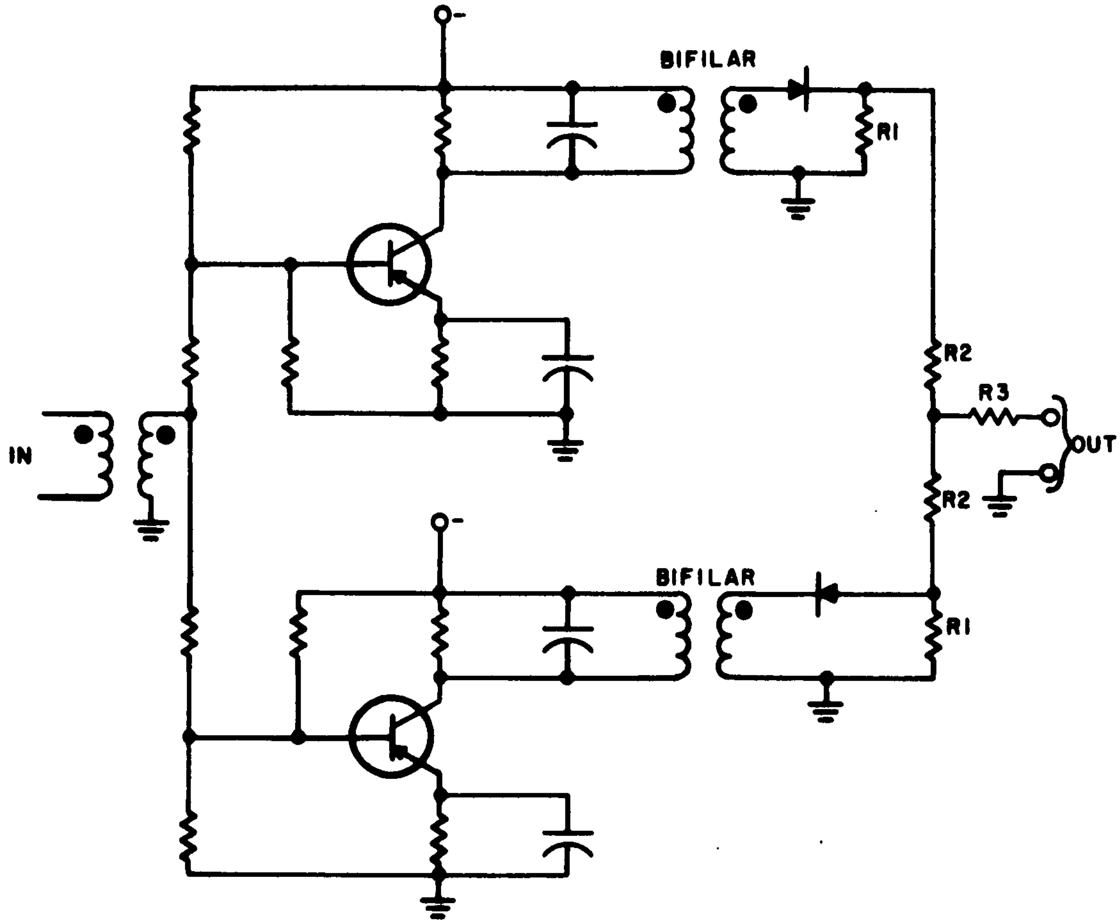

Fig. 8-1 Schematic of a Broadband Discriminator

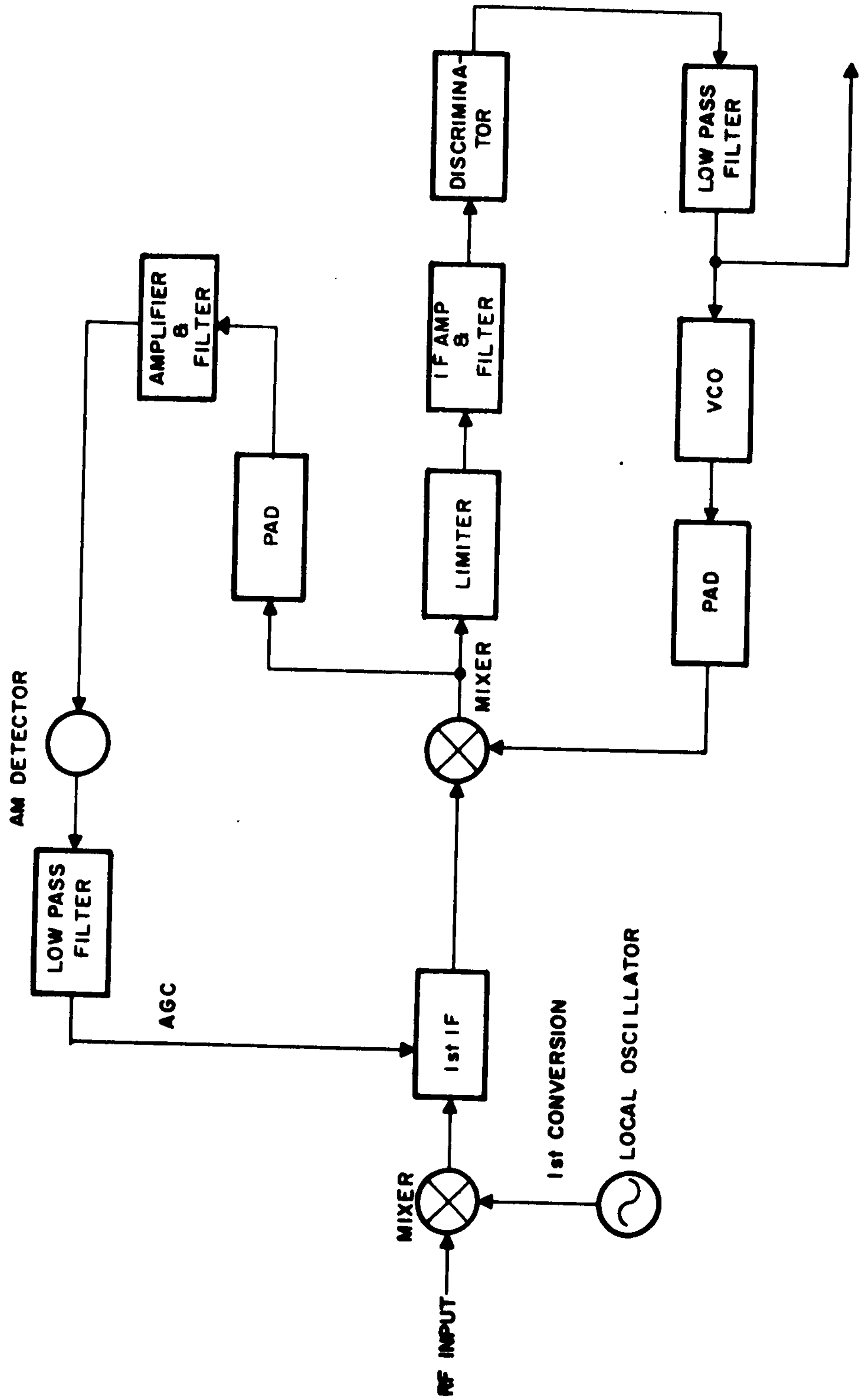

Fig. 8-2 Block Diagram of a Loop with Double Conversion and AGC

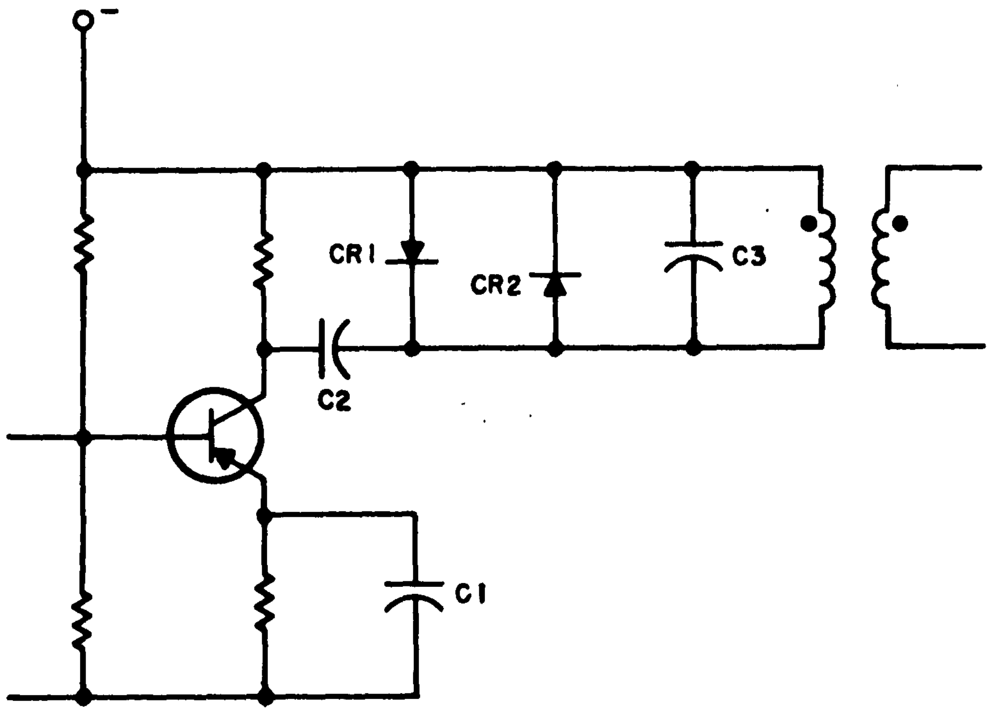

Fig. 8-3 Diagram of a Combination IF Amplifier, IF Filter (Single-Pole, Wide-Band), and a Limiter

SECTION 9

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SECTION 10

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APPENDIX A

DERIVATION OF A LINEAR MODEL

A.1 GENERAL

Because a frequency feedback discriminator operates on frequency, it is useful to "linearize" the loop in terms of frequency into an equivalent model. Explicitly, the loop components will be reduced by this "linearization" to forms that are S-plane functions of input frequency.

A system block diagram is illustrated in Section 3 (Figure 3-1). In this figure, $\omega_2/2\pi$ is the VCO center frequency; $\hat{\phi}$ is the VCO estimate of the input signal phase, and θ_2 is a phase term encompassing noise jitter as well as all VCO phase components not contained in $\omega_2 t$ and $\hat{\phi}$.

A.2 FORMULAS

The loop input in terms of phase is $\theta = \omega_c t + \phi_1 + \phi_n$, and so the equivalent frequency input is:

$$f_{in} = \frac{d\,\theta_{in}}{2\pi\,dt} = \frac{\omega_c + \frac{d\,\phi_1}{dt} + \frac{d\phi_n}{dt}}{2\pi} \qquad \text{(A-1)}$$

where $\frac{\omega_c}{2\pi}$ = the carrier frequency in cps

ϕ_1 = information phase

ϕ_n = noise phase contribution

treating ϕ_1 as sinusoidal modulation, $\phi_1 = \frac{f_d}{f_m} \cos \omega_m t$

$$\frac{d\,\phi_1}{dt} = 2\pi f_d \sin \omega_m t \qquad \text{(A-2)}$$

where f_d = carrier frequency deviation in cps

$\frac{\omega_m}{2\pi}$ = modulation frequency in cps.

In equation (A-1), ϕ_n represents the phase contribution from noise, which is assumed to be Gaussian "white" noise within a band B_{RF}. To determine ϕ_n, one must start with the incremental noise voltage which can be represented as $n(t) = A_n \cos (\omega_c + \omega)t$. The next step, the combination of signal and noise, is made by assuming the carrier is unmodulated, or $e_s = A_c \cos \omega_c t$. The sum of signal and noise is then:

$$e(t) = A_n \cos (\omega_c + \omega)t + A_c \cos \omega_c t$$

$$= A(t) \cos (\omega_c t + \phi_{n1}) \qquad \text{(A-3)}$$

where $\phi_{n1} = \tan^{-1} \left(\dfrac{A_n \sin \omega t}{A_c + A_n \cos \omega t} \right)$

If a limiter precedes the demodulator, the time-varying amplitude in equation (A-3), A(t), is replaced by a constant amplitude E. Then:

$$\phi_{n_1} = \phi_n = \tan^{-1} \left(\frac{A_n \sin \omega t}{A_c + A_n \cos \omega_c t} \right) \qquad \text{(A-4)}$$

and the frequency contribution of ϕ_n is found by differentiating equation (A-4), or:

$$\frac{1}{2\pi} \frac{d\phi_n}{dt} = \frac{\omega A_n^2 + A_c A_n \omega \cos wt}{\left(A_c^2 + A_n^2 + 2A_c A_n \cos \omega_c t\right) 2\pi} \qquad \text{(A-5)}$$

If a high input SNR is assumed; i.e., $A_c >> A_n$, equation (A-5) becomes:

$$\frac{1}{2\pi} \frac{d\phi_n}{dt} \approx \frac{A_n f}{A_c} \cos wt \qquad \text{(A-6)}$$

and now the equivalent noise power spectrum can be determined. The mean noise power input due to a noise component at some frequency f is:

$$\Delta N = \frac{f^2}{2} \left(\frac{A_n}{A_c} \right)^2 \qquad \text{(A-7)}$$

But $\frac{A_n^2}{2} = N_o \Delta f$, where N_o is the power spectral density of the incoming Gaussian white noise. Then equation (A-7) reduces to:

$$\frac{\Delta N}{\Delta f} = f^2 \frac{N_o}{A_c^2}$$

G_f (f) is defined as being equal to $\frac{\Delta N}{\Delta f}$, where G_f (f) is the equivalent noise spectral density for a frequency type input; hence:

$$G_f^{(f)} = \frac{f^2 N_o}{A_c^2} \quad \text{but } (\rho_i)_i = \frac{A_c^2}{2 N_o B_{\gamma RF}}, \text{ or therefore:}$$

$$G_f (f) = \frac{f^2}{\rho_i \; 2 B_{RF}} \qquad \rho_i = \text{input SNR.}$$

The loop input is now determined by combining equations (A-1), (A-2), and (A-5):

$$f_{in} = \frac{\omega_c}{2\pi} + f_d \sin \omega_m t$$

$$+ \frac{A_n^2 (\omega \cos^2 \omega_c t + \omega_c \sin^2 \omega_c t + A_c A_n \cos \omega t)}{2\pi (A_c^2 + A_n^2 + 2 A_c A_n \cos \omega_c t)} \qquad \text{(A-8)}$$

Equation (A-8) becomes:

$$f_{in} (t) \approx \frac{\omega_c}{2} + f_d \sin \omega_m t + \frac{A_n}{2\pi A_c} \omega \cos wt \qquad \text{(A-9)}$$

for $A_c >> A_n$

The next step is reducing the mixer shown in Figure A-1 to a linearized form. A mixer is basically a product device (multiplier), and the mixer used in the feedback discriminator multiplies the input wave with the wave generated by the VCO. Thus the mixer output is:

$$K_m A_c E_2 \cos (W_1 t + \phi_1 (t) + \phi_m) \cos (\omega_2 t + \hat{\phi} (t) + \phi_2)$$

which is in the form:

A cos α cos β, but $\cos \alpha \cos \beta = 1/2 \cos (\alpha - \beta) + 1/2 \cos (\alpha + \beta)$
Therefore, the mixer output is:

$$e_m = E_3 \left\{ \cos \left[(\omega_1 - \omega_2) t + \phi_1 - \hat{\phi} + \phi_m - \phi_2 \right] \right\}$$
$$+ E_3 \left\{ \cos \left[(\omega_1 + \omega_2) t + \phi_1 + \hat{\phi} + \phi_m + \phi_2 \right] \right\} \qquad \text{(A-10)}$$

The second half of equation (A-10) can be disregarded because its frequency components are outside the passband of the bandpass filter following the mixer (the filter passband is centered at $\frac{\omega_1 - \omega_2}{2 \pi}$).
In terms of frequency, then, the mixer output is:

$$\frac{\omega_c - \omega_2}{2 \pi} + f_d \sin w_m t - \frac{d \hat{\phi}}{2 \pi \, dt} + \frac{d \phi_m}{2 \pi \, dt} - \frac{d \phi_2}{2 \pi \, dt}$$

Consequently, the mixer becomes a frequency summer in which the loop input is considered as positive while the VCO output is taken as being negative.

The next loop component to be considered is the bandpass filter. This filter will be reduced from a bandpass to its equivalent low pass form using the method outlined by Weaver and Lawhorn (Ref. 8) in an I.R.E. letter. Although Weaver and Lawhorn performed their analysis for a phase-locked loop in which a phase detector was used instead of a frequency discriminator, a frequency discriminator in the frequency domain corresponds to a phase detector in the phase domain, and with this translation, their analysis is apropos to the feedback discriminator problem.

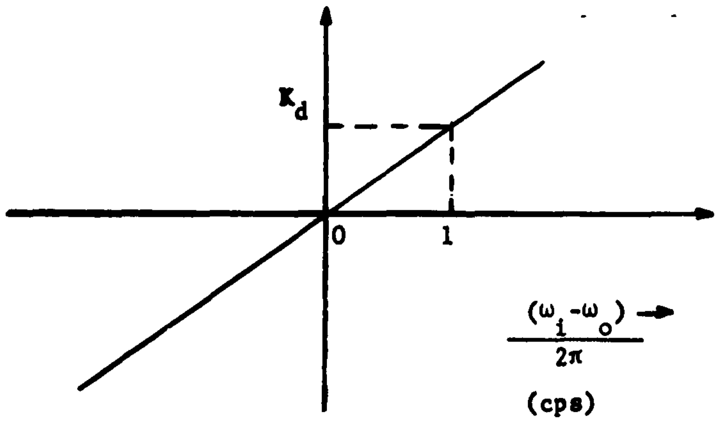
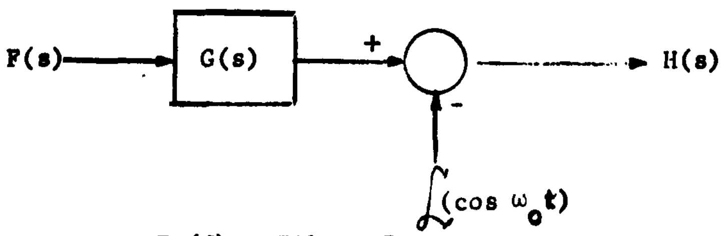


Figure A-1 Characteristics of an Ideal Discriminator

In Figure A-1, K_d is the output amplitude in volts when $\frac{\omega_i - \omega_o}{2\pi} = 1$. The transfer characteristics shown in Figure A-1 can be considered as resulting from a mixer fed by an oscillator sitting at ω_o and by the bandpass filter output. The block diagram of this circuit is shown in Figure A-2.



F (S) = Filter Input
G (S) = Filter Transform

Figure A-2 Block Diagram Showing Bandpass Filter, Reference Oscillator, and Mixer

If F (s) is the Laplace transform of the filter input, the filter output is F (s) G (s). But F(s) may be factored into $F(s) = F_x(s)\, F_x^*(s)$, where F_x (s) contains all the upper half S-plane poles and zeroes of F (s), and $F_x^*(s)$ contains all the lower half S-plane poles and zeroes of F (s). Similarly, let $G(s) = G_x(s)\, G_x^*(s)$. The Fourier transform of f(t) is the same as the Laplace transform (with S = ω) of f(t) if:

$$\int_0^{\infty} \left| f(t) \right| d\omega < \infty \text{ and } f(t) = 0 \text{ for } t < 0$$

$$\text{or } F(\omega) = F(s) \Big|_{s = \omega}$$

Similarly, $G(\omega) = G(s) \Big|_{S = \omega}$. However, the Fourier transform of

$$\cos \omega_m t \text{ is } \pi \left[\delta(\omega - \omega_m) + \delta(\omega + \omega_m) \right] \text{ because:}$$

$$F\left\{\cos \omega t\right\} \neq \mathcal{L}\left\{\cos \omega t\right\} \Big|_{\omega = s}$$

due to the fact that $\int_{-\infty}^{\infty} \left|\cos \omega t\right| d\omega \nless \infty$. $\delta(\omega)$ is the impulse function.

The Fourier transform of h(t) is the convolution of the individual transforms of f(t), G(t), and the reference cosine wave, or:

$$H(\omega) = \pi \int_{-\infty}^{\infty} F_x(\omega - \omega_o)\, F_x^*(\omega - \omega_o) \tag{A-11}$$

$$G_x^*(\omega - \omega_o) \left[\delta(\omega - \omega_o) + \delta(\omega + \omega_o) \right] d\omega$$

Integrating, equation (A-11) becomes:

$$H(\omega) = \pi \left[F_x(\omega - \omega_o)\, F_x^*(\omega - \omega_o)\, G_x(\omega - \omega_o)\, G_x^*(\omega - \omega_o) \right] \tag{A-12}$$

$$+ F_x(\omega + \omega_o)\, F_x^*(\omega + \omega_o)\, G_x(\omega + \omega_o)\, G_x^*(\omega + \omega_o)$$

At this point, it is necessary to make an assumption about ω_o, the assumption being that the center of the bandpass filter is selected so

that it is ω_c and the center frequency of f(t) is ω_c, or in the words $F_x(\omega)$ and $G_x(\omega)$ are symmetrical about ω_c. When this is true;

$$F_x(\omega - \omega_o) = F_x^*(\omega + \omega_o) \text{ and } G_x(\omega - \omega_o) = G_x^*(\omega + \omega_o)$$

The term $F_x^*(\omega - \omega_o)$ represents the moving of the $F_x(\omega)$ lower plane pole zero center point from $-\omega_o$ to $-2\,\omega_o$, while $F_x(\omega + \omega_o)$ moves the $F_x(\omega)$ pole and zero center from ω_o to $2\,\omega_o$. Therefore $F_x^*(\omega - \omega_o) = F_x(\omega + \omega_o)$, and the same reasoning applies to $G_x(\omega + \omega_o)$ and $G_x^*(\omega - \omega_o)$. However, these high frequency components contributed by $G_x(\omega + \omega_o)$ and $F_x(\omega + \omega_o)$ can be lumped together as simply a gain constant, say K_H, or:

$$K_H = G_x(\omega + \omega_o)\; F_x(\omega + \omega_o)$$

$$K_H^* = G_x^*(\omega - \omega_o)\; F_x(\omega - \omega_o)$$

Equation (A-12) then reduces to:

$$H(\omega) = K_H^* F_x\, x(\omega - \omega_o)\; G_x(\omega - \omega_o) + K_H F_x(\omega - \omega_o)\, G_x(\omega - \omega_o)$$

From complex variable theory:

$$Z + Z^* = 2\,\text{Re}(Z) \text{ or:}$$

$$H(\omega) = 2\,\text{Re}(K_H)\; x\; F_x(\omega - \omega_o)\; G_x(\omega - \omega_o) \qquad \text{(A-13)}$$

In Laplace domain:

$$H(s)\; \alpha\; F_x(S - j\,\omega_o)\; G_x(S - j\,\omega_o)$$

Hence, the filter is translated by $j\omega_o$ so that its pole zero pattern is centered about the origin instead of ω_o, and the bandpass filter is now equivalent to a low pass filter.

$F_x^*(\omega)$ and $F_x(\omega)$ are also translated to the origin, but $F_x(\omega)\ F_x^*(\omega)$ is the transform of f(t). Hence the input to the bandpass can be treated as a low frequency input, and this same f(t) is also the output of the mixer shown in Figure 3-1. The "linearized" mixer output is therefore

$$f(t) = f_d \sin \omega_m t - \frac{d\hat{\phi}}{2\pi\, dt} + \frac{d\phi_n}{2\pi\, dt} - \frac{d\phi_2}{2\pi\, dt} \tag{A-14}$$

Referring to Figure A-1, the discriminator transfer function is seen to be a constant K_d times the frequency input (bearing in mind the frequency transformation $\frac{\omega_i - \omega_o}{2\pi}$ which was previously justified).

The VCO shown in Figure 3-1 has the characteristic that its output frequency changes in a linear manner with a change in input voltage. Therefore, the VCO assumes a constant, say K_v, as its transfer function in the frequency domain.

All the loop components have now been reduced to their Laplace or S-plane forms for a frequency input. This information is summarized in Figure A-3, which shows this "linearized" loop in block diagram form.

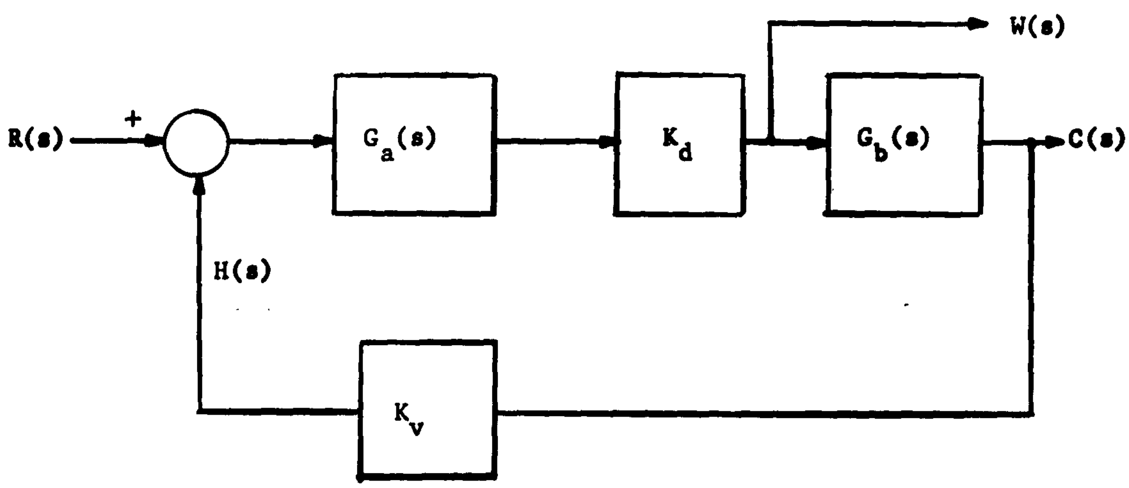


Figure A-3 Linearized Loop

In Figure A-3, $H(s) = \mathcal{L}\left(\frac{d\hat{\phi}}{2\pi\ dt} + \frac{d\phi_2}{2\pi\ dt}\right)$ and:

$$R(s) = \mathcal{L}\left(f_d \sin \omega_m t + \frac{d\phi_m}{2\pi\ dt}\right)$$

APPENDIX B

TRANSFER FUNCTIONS

B.1 OPTIMIZATION OF TRANSFER FUNCTIONS

When one speaks of transfer functions, the logical query is: "How is the transfer function optimized?" Unfortunately, there is no clear-cut answer since a universal optimum criterion has not been established. For example, one could use the Weiner-Hopf criterion to optimize SNR or one of several servo theory criteria to optimize loop time response.

The Weiner optimum transfer function is predicated on the assumptions that the system is linear, the process is stationary, and the mean square error is an appropriate measure of the system error (Ref. 1). When these assumptions hold, the following equation is valid:

$$G_o(s) = \frac{1}{\Phi_{ii}^{+}(s)} \mathcal{L}\mathcal{F}^{-1} \left[\frac{\Phi_{ss}(s)\ G_d(s)}{\Phi_{ii}^{-}(s)} \right] \qquad \text{(B-1)}$$

where $\mathcal{L}$ = Laplace transform

$\mathcal{F}^{-1}$ = Inverse Fourier transform

$G_o(s)$ = Weiner optimum filter transfer function

$G_d(s)$ = Desired filter transfer function

$\Phi_{ss}(s)$ = Signal power spectral density

$\Phi_{nn}(s)$ = Noise power spectral density

$\Phi_{ii}(s) = \Phi_{ss}(s) + \Phi_{nn}(s)$

$\Phi_{ii}(s) = \Phi_{ii}^{+}(s) + \Phi_{ii}^{-}(s)$

Equation (B-1) gives the optimum transfer function in terms of SNR according to the Weiner-Hopf criterion. However, a specific solution of equation (B-1) is good only for the signal and noise spectra used in the solution, and the solving of equation (B-1) is extremely difficult for certain signal spectra such as a step function.

Moreover, it is desirable to optimize the loop performance for a step input since this is a commonly encountered stimulus. Consequently, criteria in addition to the Weiner-Hopf optimizing criterion merit examination, and several servo theory criteria will be considered (Ref. 2). Assuming a loop transfer function of the form given by equation (B-2), the Butterworth optimizing function, which locates the denominator poles on a semicircle (the center being at the S-plane origin) in the left half-plane.

$$\frac{C(s)}{R(s)} = \frac{C}{S^{V} + b_{v-1} S^{V-1} + \ldots + b_2 s^2 + b_1 s + n^2} \qquad \text{(B-2)}$$

The Butterworth denominator coefficients for values of V from one to five are given in Table B-1.

TABLE B-1

BUTTERWORTH OPTIMIZING FORM

$S + \omega_n$
$S^2 + 1.4\omega_n S + \omega_n^2$
$S^3 + 2\ \omega_n S^2 + 2\ \omega_n S^2 + \omega_n^3$
$S^4 + 2.6\ \omega_n S^3 + 3.4\ \omega_n^2 S^2 + 2.6\ \omega_n^3 S + \omega_n^4$
$S^5 + 3.24\ \omega_n S^4 + 5.29\ \omega_n^2 S^3 + 5.24\ \omega_n^3 S^2 + 3.24\ \omega_n^4 S + \omega_n^5$

Another criterion deserving consideration is the ITAE (integral of time multiplied by the absolute value of error) criterion. With the ITAE method, the integral

$$I = \int_0^\infty t \left|\epsilon\right| dt \text{ is minimized.}$$

When the transfer function has the form of equation (B-2), Table B-2 gives the ITAE optimum denominator coefficients for values of V from one to five.

TABLE B-2

ITAE OPTIMIZING FORM

$S + \omega_n$
$S^2 + 1.4\ \omega_n S + \omega_n^2$
$S^3 + 1.75\ \omega_n S^2 + 2.15\ \omega_n^3 S + \omega_n^3$
$S^4 + 2.1\ \omega_n S^3 + 3.5\ \omega_n^2 S^2 + 2.7\ \omega_n^3 S + \omega_n^4$
$S^5 + 2.8\ \omega_n S^4 + 5\ \omega_n^2 S^3 + 5.5\ \omega_n^3 S^2 + 3.4\ \omega_n^4 S + \omega_n^5$

Both Tables B-1 and B-2 have been tabulated for values of V from one to five, and since the two tables were derived from the same transfer function form, i.e., equation (B-2), a comparison between them can be made. The optimizing forms are identical when V = 1 or 2.

While there is some difference between tables for V = 3, 4, and 5, this difference is not very great, and it can be neglected in the design of feedback discriminators. Consequently, one can use either Table B-1 or B-2 to design the feedback loop.

B.2 LOOP TRANSFER FUNCTIONS FOR THE FEEDBACK DISCRIMINATOR

Using the linearized loop form shown in Figure A-4 of Appendix A, the loop transfer functions at the VCO output, discriminator output, mixer output, and low pass filter output can be ascertained. Loop

transfer for the general case as portrayed by Figure A-4 and the particular case when $G_b(s) = \frac{K_{ab}}{s + a}$ and $G_a(s) = \frac{a}{s + a}$ will be determined.

The transfer equation for the low pass filter output is:

$$\frac{C(s)}{R(s)} = \frac{G_a(s)\ G_b(s)\ K_d}{1 + G_a(s)\ G_b(s)\ K_d\ K_v} \tag{B-3}$$

when: $G_a(s) = \frac{a}{S + a}$ and $G_b(s) = \frac{K_A\ b}{S + b}$

$$\frac{C(s)}{R(s)} = \frac{a\ b\ K_d\ K_A}{S^2 + 2\zeta\ \omega_n S + \omega_n^2} \tag{B-4}$$

at the VCO output:

$$\frac{H(s)}{R(s)} = \frac{G_a(s)\ G_b(s)\ K_d\ K_v}{1 + G_a(s)\ G_a(s)\ K_d\ K_v} \tag{B-5}$$

Or, in the specific case:

$$\frac{H(s)}{R(s)} = \frac{\omega_n^2 - a\ b}{S^2 + 2\zeta\ \omega_n S + \omega_n^2} \tag{B-6}$$

the discriminator output is:

$$\frac{W(s)}{R(s)} = \frac{G_a(s)\ K_d}{1 + G_a(s)\ G_b(s)\ K_d\ K_v} \tag{B-7}$$

for the specific case:

$$\frac{W(s)}{R(s)} = \frac{(S + b)\ a K_d}{S^2 + 2\zeta\ \omega_n S + \omega_n^2} \tag{B-8}$$

Next, the mixer output, which is the error E (s), will be found:

$$\frac{E(s)}{R(s)} = \left[1 - K_v\, C(s)\right] \tag{B-9}$$

$$= \frac{1}{1 + G_a(s)\; G_b(s)\; K_d\; K_v}$$

When $\quad G_b(s) = \dfrac{K_A\, b}{S + b} \quad$ and $\quad G_a(s) = \dfrac{a}{S + a}$

$$\frac{E(s)}{R(s)} = \frac{S^2 + 2\zeta\ \omega_n S + a\ b}{S^2 + 2\zeta\ \omega_n S + \omega_n^2} \tag{B-10}$$

Equations (B-3) through (B-10) are summarized in Table 4-1 of Section 4.

B.3 IF TRANSFER FUNCTION CONSIDERATIONS

The IF filter in a feedback discriminator is tuned to the center IF frequency. Hence, the filter is tuned when the input carrier is unmodulated (there is a tacit assumption that the frequency difference between the carrier and the VCO center frequency is equal to the center IF frequency so that there is no frequency error). But modulation, which varies much more slowly than the carrier, has the effect of detuning the IF filter (Ref. 3) with respect to the IF frequency (the detuning is maximum at modulation peaks).

This detuning has a deleterious effect on the loop because it increases phase shift in the IF filter. The increased phase shift in turn decreases the loop phase margin. In fact, the loop will oscillate on modulation peaks if the filter phase shift is great enough.

In order to minimize phase shift produced by modulation detuning in the IF filter, it has been shown (Ref. 3) that a single-pole "slow roll-off" filter should be used. The term "slow roll-off" is arbitrary, but

in this context it is meant to denote an attenuation slope of 6 db per octave, or less outside the passband. Consequently, the IF filter has an equivalent low pass transfer function whose form is $\frac{b}{S + b}$.

B.4 CONSIDERATION OF TRANSFER FUNCTION ORDER

The highest order of the loop transfer function denominator at the summing junction is equal to the number of loop poles because there must be at least as many poles as zeros. On the other hand, the highest order of the loop transfer function numerator at the summing junction is equal to the number of loop zeros. The preceding statements were deduced from equation (B-11) which was derived from Figure B-1.

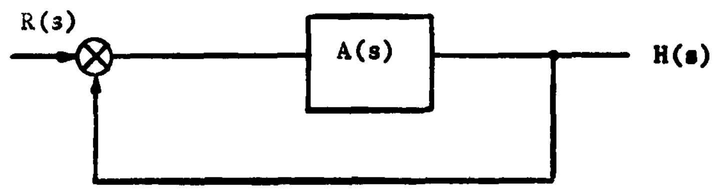


Figure B-1 Diagram Showing H(s) as the Loop Transfer Function at the Summing Junction

Let $A(s) = \frac{N(s)}{D(s)}$ where the zeros of N(s) are the zeros of A(s) while the zeros of D(s) are the poles of A(s). Moreover:

$$H(s) = \frac{A(s)}{1 + A(s)} \quad \text{or} \quad H(s) = \frac{N(s)}{D(s) + N(s)} \qquad \text{(B-11)}$$

Recalling from elementary theory the fact that each pole of A(s) contributes a -90° phase shift at $\omega = \infty$, while each zero contributes a $+90^\circ$ phase shift at $\omega = \infty$, one sees that the phase margin is reduced relatively to the increase in poles.

Consequently, there is a possibility the loop will oscillate if there are more than two poles, and care must be exercised in the design to ensure that there is sufficient phase margin at the gain cross-over. Another attribute of poles is the effect on response time. Generally, increasing the number of loop poles increases the loop time response and thereby has a deleterious effect on loop transient response (Ref. 2).

Considering the undesirable effects produced by multi-poles; i.e., reduction of phase margin and lengthening of response time, the best design for a feedback discriminator seems to be one incorporating two poles with no zeros; one pole is contained in the IF filter, while the other is contained in the low pass filter at the discriminator output. The resultant function is then in the form of equation (B-4).

APPENDIX C

TRANSIENT RESPONSE

C.1 TRANSIENT RESPONSE FOR SINE WAVE INPUT

The transient response of a feedback discriminator is useful to know because the loop transient response defines the loop dynamic behavior in the presence of an input stimulus. For example, one can predict the accuracy with which the loop tracks the stimulus and the tracking range by determining the transient error, denoted ϵ (t).

While many loop configurations are possible, the configuration chosen for this analysis is the loop that contains, when reduced to its linear equivalent, a single-pole bandpass IF filter and a single-pole baseband filter as shown in Figure C-1. However, the analysis method used in examining this loop configuration can be extended to any other loop form.

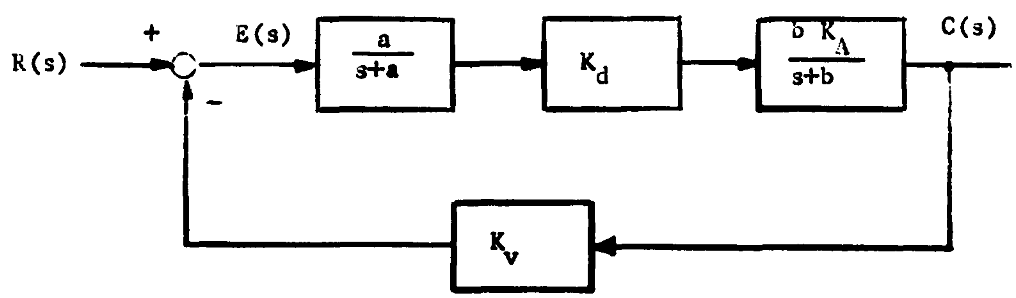


Figure C-1. Loop Form used in the Error Transient Analysis

When the input to the loop, which is applied at time $t = 0$, is a carrier being FM-modulated by a sine wave, the time-varying signal is:

$$e_s (t) = A_c \sin (\omega_c t + \frac{f_d \cos \omega_m t}{f_m})$$

or in terms of frequency:

$$f_{in} = \frac{\omega_c}{2\pi} + f_d \sin \omega_{mt} \quad \text{since} \quad f_{in} = \frac{d}{dt}\frac{\theta_{in}}{2\pi}$$

However, the center frequency of the linearized loop (see Appendix A) is ω_c, and therefore the frequency input can be taken as:

$$f_1(t) = f_d \sin \omega_{mt} \tag{C-1}$$

In terms of the S-domain, equation (C-1) becomes:

$$F_1(s) = \frac{f_d \omega_m}{S^2 + \omega_m^{\,2}} \tag{C-2}$$

Examining the servo loop illustrated in Figure C-1, one finds that the S-domain error, E(s), is:

$$E(s) = R(s) \left[1 - C(s) \quad K_v \right] \tag{C-3}$$

On the other hand:

$$\frac{C(s)}{R(s)} = \frac{a\, K_d\, K_a\, b}{S^2 + S\,(a + b) + ab\,(1 + K_A\, K_d\, K_v)} \tag{C-4}$$

Combining equations (C-3) and (C-4):

$$E(s) = R(s) \frac{S^2 + S\,(a + b) + ab}{S^2 + S\,(a + b) + ab\,(1 + K_A\, K_d\, K_v)} \tag{C-5}$$

To simplify equation (C-5), make the following substitutions:

$$\omega_n^{\ 2} = ab\ (1 + K_A\ K_d\ K_v)$$

$$2\zeta\omega_n = a + b$$

Equation (C-5) then becomes:

$$E(s) = R(s) \left(\frac{S^2 + 2\zeta\omega_n S + ab}{S^2 + 2\zeta\omega_n S + \omega_n^{\ 2}} \right) \qquad \text{(C-6)}$$

From equation (C-2), $R(s) = \dfrac{f_d\ \omega_m}{S^2 + \omega_m^{\ 2}}$ since:

$$R(s) = F_1(s) = \frac{f_d\omega_m}{S^2 + \omega_m^{\ 2}}$$

In the time domain;

$$\epsilon(t) = f_d\omega_m\ \mathcal{F}^{-1} \left[\frac{S^2 + 2\zeta\omega_n S + ab}{(S^2 + \omega_m^{\ 2})(S^2 + 2\zeta\omega_n S + \omega_n^{\ 2})} \right] \qquad \text{(C-7)}$$

Performing the indicated operation on equation (C-7) yields:

$$\epsilon(t) = f_d \omega_m \left[\frac{1}{\omega_m} \sqrt{\frac{4\zeta^2 \omega_n^2 \omega_m^2 + (ab - \omega_m^2)^2}{4\zeta^2 \omega_n^2 \omega_m^2 + (\omega_n^2 - \omega_m^2)^2}} \sin(\omega_m t + \psi_1) \right.$$

$$+ \frac{e^{-\zeta\omega_n t}}{\omega_n \sqrt{1 - \zeta^2}} \sqrt{\frac{(ab - \omega_n^2)^2}{4\zeta^2 \omega_n^2 \omega_m^2 + (\omega_n^2 - \omega_m^2)^2}}$$

$$\left. \sin\left(\omega_n \sqrt{1 - \zeta^2}\, t + \psi_2\right) \right] \quad \text{in cps}$$

where:

$$\psi_1 = \tan^{-1}\left(\frac{2\zeta\omega_n\omega_m}{ab - \omega_m^2}\right) - \tan^{-1}\left(\frac{2\zeta\omega_n\omega_m}{\omega_n^2 - \omega_m^2}\right)$$

$$\psi_2 = \tan^{-1}\left[\frac{2\zeta\omega_n^2\sqrt{1 - \zeta^2}}{\omega_n^2 (2\zeta^2 - 1) + \omega_m^2}\right] \qquad \text{(C-8)}$$

C.2 TRANSIENT RESPONSE FOR A FREQUENCY STEP

The next input stimulus to be considered will be a frequency step, and the transient error will be found for the loop configuration depicted in Figure C-1. For a frequency step, the incoming time-varying signal voltage is:

$$\rho_s(t) = \left[A_c \sin \omega_c t + \mu_2(t)\, \omega_x t\right]$$

where:

$$\mu_2(t) = 1 \text{ for } t \geq 0$$

$$\mu_2(t) = 0 \text{ for } t < 0 \qquad \text{(C-9)}$$

In terms of frequency input for $t > 0$, $f_1(t) = \frac{\omega_x}{2\pi}$, and:

$$F_1(s) = \frac{\omega_x}{2\pi s} \qquad \text{(C-10)}$$

Substituting equation (C-10) as R(s) into equation (C-6) yields an error of:

$$E(s) = \frac{\omega_x}{2\pi s} \frac{S^2 + 2\zeta\omega_n S + ab}{S^2 + 2\zeta\omega_n S + \omega_n{}^2} \qquad \text{(C-11)}$$

In terms of time, equation (C-11) becomes:

$$\epsilon(t) \mathcal{F}^{-1} \frac{\omega_x}{2\pi s} \frac{S^2 + 2\zeta\omega_n S + ab}{S^2 + 2\zeta\omega_n S + \omega_n{}^2} \qquad \text{(C-12)}$$

Performing the indicating operation on the right side of equation (C-12) gives:

$$\epsilon(t) = \frac{\omega_x}{2\pi} \left[\frac{ab}{\omega_n{}^2} - \frac{(ab - \omega_n{}^2)}{\omega_n{}^2 \sqrt{1 - \zeta^2}} \rho^{-\zeta\omega_n t} \sin \left(\omega_n \sqrt{1 - \zeta^2}\, t + \psi \right) \right]$$

in cps

where:

$$\psi = \tan^{-1} \sqrt{\frac{1 - \zeta^2}{\zeta^2}} \qquad \text{(C-13)}$$

The loop error transient response for a sine wave input was derived in Section C-1 with the end result given by equation (C-8). Additionally, the loop error transient response for a frequency step input was found in Section C-2, and equation (C-13) displays this result.

APPENDIX D

DERIVATION OF CLOSED LOOP NOISE BANDWIDTH

D.1 GENERAL

The closed loop noise bandwidth; i.e., the noise bandwidth at the VCO output, plays an important role in the loop design (described in Appendix F); it is therefore necessary to derive an expression relating the closed loop noise bandwidth to the other loop parameters. The closed loop noise bandwidth will be found as an equivalent rectangular band for a flat spectrum input. This equivalent bandwidth is given by the following well-known equation:

$$B_{nl} = \frac{1}{2\pi H_m^2} \int_{-\infty}^{\infty} \left| \frac{H(jw)}{R(jw)} \right|^2 dw \tag{D-1}$$

where:

H_m = the maximum absolute value of $\frac{H(jw)}{R(jw)}$

$\frac{H(jw)}{R(jw)}$ = the transfer function at the VCO output

To determine the noise bandwidth at the VCO output, Figure D-1, the transfer function for the VCO output is $\frac{H(s)}{R(s)}$, and:

$$\frac{H(s)}{R(s)} = \frac{K_v K_d K_A ab}{S^2 + (a + b) S + ab (1 + K_v K_d K_A)} \tag{D-2}$$

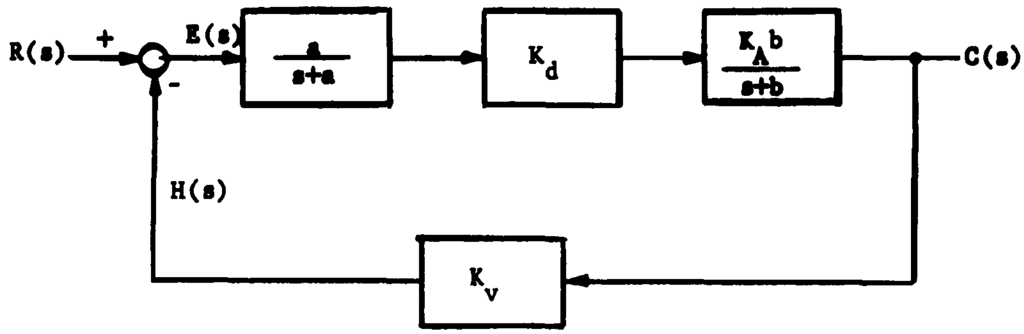


Figure D-1. Circuit whose Closed Loop Noise Bandwidth will be Determined.

If the following substitutions are made:

$$a + b = 2\zeta\omega_n$$

$$ab\,(1 + K_v K_d K_A) = \omega_n^2$$

Equation (D-2) becomes:

$$\frac{H(s)}{R(s)} = \frac{K_v K_d K_A\, ab}{S^2 + 2\zeta\omega_n S + \omega_n^2} \qquad \text{(D-3)}$$

Letting $S = jw$ in equation (D-3) yields:

$$\frac{H(jw)}{R(jw)} = \frac{K_v K_d K_A\, ab}{-\omega^2 + 2\,j\zeta\omega_n\,\omega + \omega_n^2} \qquad \text{(D-4)}$$

The maximum absolute value of $\frac{H(jw)}{R(jw)}$ in equation (D-4) is clearly:

$$H_m = \frac{K_v K_d K_A}{1 + K_v K_d K_A} \qquad \text{(D-5)}$$

Substituting equations (D-4) and (D-5) into equation (D-1) gives:

$$B_{nl} = \frac{(1 + K_v\ K_d\ K_A)^2}{2\pi} \int_{-\infty}^{\infty} \frac{a^2 b^2\ d\omega}{\left[- \omega^2 + 2j\zeta\omega_n\omega + \omega_n^2 \right]^2} \qquad \text{(D-6)}$$

or letting I stand for the integral:

$$I = \int_{-\infty}^{\infty} \frac{d\omega}{\omega^4 + \omega^2 \left(4\zeta^2\omega_n^2 - 2\omega_n^2 \right) + \omega_n^4} \qquad \text{(D-7)}$$

The next step is to complete the square in the denominator of equation (D-7) so that the denominator can be factored:

$$I = \int_{-\infty}^{\infty} \frac{d\omega}{\left[\omega^4 + \omega^2 \left(4\zeta^2\omega_n^2 - 2\omega_n^2 \right) + \omega_n^4 \right] - \frac{16\zeta^4\ \omega_n^4}{4}}$$

$$+ 4\zeta^2\omega_n^4 - \omega_n^2 + \omega_n^4$$

and;

$$I = \int_{-\infty}^{\infty} \frac{d\omega}{\left[\omega^2 + \frac{\omega_n^2 (4\zeta^2 - 2)}{2} \right] - \frac{4\zeta^2\omega_n^4 (4\zeta^2 - 4)}{4}}$$

so:

$$I = \int_{-\infty}^{\infty} \frac{d\omega}{\left[\omega^2 + \omega_n^2 \; (2\zeta^2 - 1) + 2j\zeta\sqrt{1 - \zeta^2}\right] \left[\omega^2 + \omega_n^2 \; (2\zeta^2 - 1) - 2j\zeta \;\; \omega_n^2\sqrt{1 - \zeta^2}\right]}$$

The definite integral in equation (D-8) can be evaluated by contour integration, as shown by the Residue Theorem of complex variable theory. The Residue Theorem is stated as follows:

Let 1 be a closed curve on and within which $g(z)$ is analytic with the exception of a set of finite singularity points $(Z_1, Z_2, \ldots, Z_n)$ enclosed by 1.

If $K_1, K_2, \ldots\ldots, K_n$ represent the residues of $g(z)$ at these singularity points, then:

$$\int_I g(z)\, dz = 2\pi j \, (K_1 + K_2 + \ldots + K_n) \qquad \text{(D-9)}$$

where the integration is taken in a counterclockwise sense around 1.

The integral path of the complex function in equation (D-8) is taken to be the real axis from $+\infty$ to $-\infty$ with a semicircular curve at $|Z| = \infty$ in the upper half-plane joining these ends of the real axis.

To evaluate the integral of equation (D-8) through the use of equation (D-9) in a straightforward manner, the denominator of equation (D-8) is factored to facilitate finding the residues.

$$\left[\omega^2 + \omega_n^2 (2\zeta^2 - 1) + 2\zeta\omega_n^2 j\sqrt{1 - \zeta^2}\right]\left[\omega^2 + \omega_n^2 (2\zeta^2 - 1) - 2\zeta\omega_n^2\sqrt{1 - \zeta^2}\right]$$

$$= \frac{1}{(\omega - a_1)(\omega - a_2)(\omega - a_3)(\omega - a_4)} \qquad \text{(D-10)}$$

where:

$$a_1 = \alpha_o - j\beta_o$$

$$a_2 = \alpha_o + j\beta_o$$

$$a_3 = \alpha_o + j\beta_o$$

$$a_4 = \alpha_o - j\beta_o$$

and:

$$\beta_o = \zeta\omega_n$$

$$\alpha_o = \omega_n\sqrt{1 - \zeta^2}$$

Evaluating the upper half-plane residues of equation (D-10) yields:

$$K_2 = \frac{1}{(a_2 - a_1)(a_2 - a_3)(a_2 - a_4)}$$

$$= \frac{1}{j8\alpha_o\beta_o\ (\alpha_o - j\beta_o)}$$

$$K_3 = \frac{1}{(a_3 - a_1)(a_3 - a_2)(a_3 - a_4)}$$

$$= \frac{1}{j8\alpha_o\beta_o\ (\alpha_o + j\beta_o)}$$

But: $\quad I = 2\pi j\ (K_2 + K_3) \quad$ and therefore:

$$I = \frac{\pi}{2\ \beta_o\ (\alpha_o^{\ 2} + \beta_o^{\ 2})}$$

or: $$I = \frac{\pi}{2\zeta\omega_n^{\ 3}} \qquad \text{(D-11)}$$

Substituting the value for I given by equation (D-11) into equation (D-6) gives:

$$B_{nl} = \frac{(1 + K_v\ K_d\ K_A)^2\ \ a^2\ b^2}{2\pi} \times \frac{\pi}{2\zeta\omega_n^{\ 2}}$$

or since;

$$(1 + K_v K_d K_A)^2 \quad a^2 b^2 = \omega_n^4$$

$$B_{nl} = \frac{\omega_n}{4\zeta} \qquad \text{in cps}$$

and: $$\omega_{nl} = \frac{\pi}{2} \frac{\omega_n}{\zeta} \qquad \text{in rad/sec} \qquad \text{(D-12)}$$

Surprisingly enough, the rather formidable expression for closed loop noise bandwidth appearing in equation (D-6) reduced to a simple relation between this bandwidth, the natural frequency, and the loop damping as is demonstrated by equation (D-12).

APPENDIX E
THRESHOLDS

E.1 CLOSED LOOP THRESHOLD

The selection of a threshold point is generally a difficult task due to the arbitrariness associated with the selection. For purposes of this discussion, the closed loop threshold will be designated as the point at which a minute degradation in input SNR will produce the onset of spurious noise or "crackling" at the output of the loop. This is the same threshold definition used by Enloe (Ref. 3).

The problem is now defined as that of finding a mathematical expression which relates the threshold with the loop parameters and some measureable quantity. A logical starting point in the quest is to find the effective SNR into the loop. This is simply (Ref. 5):

$$\rho_i = \frac{A_c^2/2}{N_o B_{NRF}} \qquad \text{(E-1)}$$

where

A_c = peak signal voltage

N_o = noise power spectral density

B_{NRF} = noise bandwidth (double-sided)

$$\rho_i = \frac{1}{2\, N_o'\, B_{NRF}} \qquad \text{(E-2)}$$

where

N_o' = normalized noise power spectral density.

$N_o' = N_o/A_c^2$

As far as the loop is concerned, only the noise within the loop noise bandwidth has any effect, and thenoise bandwidth B_{nl} is given by equation (D-1) in Appendix D; i.e.,

$$B_{nl} = \frac{1}{2\pi H_m^2} \int_{-\infty}^{\infty} \left|\frac{H\ (jw)}{R\ (jw)}\right|^2 dw$$

where

H_m = the maximum absolute value of H (jw)

H(jw) = VCO output transfer function.

The equivalent input SNR from equation (E-2) is then in bandwidth B''_{nl}:

$$\rho_i = \frac{1}{2\ N''_o B_{nl}} \qquad \text{(E-3)}$$

Moreover, the rms phase error at the VCO output is:

$$\phi^2 rms = \frac{N''_o}{2\pi} \int_{-\infty}^{\infty} \frac{H\ (jw)}{R\ (jw)}\ {}^2 dw$$

$$\text{or;}\quad N''_o = \frac{2\pi\ \phi^2 rms}{\int_{-\infty}^{\infty} \frac{H\ (jw)}{R\ (jw)}\ {}^2 dw} \qquad \text{(E-4)}$$

Substituting the right side of equation (E-4) for N''_o into equation (E-3) and simultaneously replacing B_{nl} in equation (E-3) with the right side of equation (D-1) yields:

$$\rho_i = \frac{1}{2\ \phi^2_{rms}}\ H_m^2 \qquad \text{(E-5)}$$

In terms of loop parameters, H_m is given by equation (D-5), i.e., $H = \frac{K_v\ K_d\ K_a}{1+K_v\ K_d\ K_a}$. If one defines F, which is known as the feedback factor, so that $1 + K_v\ K_d\ K_a = F$, then H_m becomes:

$$H_m = \frac{F-1}{F} \qquad \text{(E-6)}$$

Placing the equality expressed expressed by equation (E-6) into equation (E-6) gives the following result:

$$\rho_i = \frac{1}{2\phi^2_{rms}} \left(\frac{F - 1}{F}\right)^2 \qquad (E-7)$$

Enloe's experiments (Ref. 3) on feedback discriminators indicate that spurious noise appears at the output of the loop when $\phi_{rms} \approx \frac{1}{3.11}$ radius. Using his results along with equation (E-7), the closed loop threshold SNR, ρ_{it}, is:

$$\rho_{iT} \approx 4.8 \left(\frac{F - 1}{F}\right)^2 \qquad (E-8)$$

Equation (E-8) is useful in designing and analyzing feedback discriminators because it explicitly relates input closed loop threshold SNR to the feedback factor F.

E.2 OPEN LOOP THRESHOLD

In addition to contending with a closed loop threshold, the designer must cope with an open loop threshold produced by the discriminator. In fact, the SNR into the discriminator must be sufficient to ensure that the discriminator is operating above its threshold, or the closed loop threshold considerations analyzed in Section E.1 are invalid and the open loop threshold holds sway over the loop operation.

The discriminator threshold, or the open loop threshold, is defined as being the point at which any decrease of discriminator input SNR will cause the output SNR to be significantly less than the output SNR predicted by the standard FM improvement formula; i.e.,

$$\rho_o = 3\,\beta^2 \; \frac{B_{NIF}}{2\,f_m} \; \rho_i \qquad (E-9)$$

where β = modulation index

B_{NIF} = IF noise band width (low pass equivalent)

f_m = highest modulation frequency

Stumpers (Ref. 6) has published discriminator curves from which one can ascertain the threshold. In his curves, Stumpers has plotted input noise-to-signal ratio vs. output noise energy for various ratios of IF bandwidth to modulation frequency. (Figure 3 in Stumper's article can be applied to any IF filter configuration if $\Delta \omega$ is taken to be the noise bandwidth of the filter).

Discriminator threshold for several ratios of IF noise bandwidth to modulation frequency are given in Table E-1.

To ascertain whether open loop threshold is satisfied for a particular loop design, the closed loop threshold is determined from equation (E-8), and the noise bandwidth of the loop is also calculated. (See Appendix D.) Obviously, the minimum SNR applied to the discriminator will be at the time when the loop SNR input is ρ_{iT}, assuming the loop is operated above the threshold.

Moreover, the noise density producing a SNR or ρ_{iT} in a noise bandwidth of B_{nl} will give a SNR of $\rho_{iT} \dfrac{B_{nl}}{B}$ in a noise bandwidth B. Consequently, the SNR at the input to the discriminator is:

$$\rho_{i0} = \rho_{iT} \frac{B_{nl}}{B_{NIF}} \qquad \text{(E-10)}$$

where

ρ_{i0} = discriminator input SNR

Expressed in decibels, equation (E-10) is:

$$10 \log_{10} \rho_{i0} = 10 \log_{10} \left(\rho_{iT} \frac{B_{nl}}{B_{NIF}} \right) \qquad \text{(E-11)}$$

Equation (E-11) is evaluated, and this result is compared with the appropriate entry in Table E-1, i.e., for the proper ratio of IF noise bandwidth to highest modulation frequency. (Extrapolation can be used for in-between values.) If the SNR in decibels obtained from equation (E-11) exceeds the threshold value from Table E-1, the open loop threshold has been satisfied; if not, the open loop threshold has not been satisfied, and the actual closed loop threshold will not be the threshold predicted by equation (E-8).

TABLE E-1

VALUES OF DISCRIMINATOR INPUT SNR AT THRESHOLD FOR VARIOUS RATIOS OF B_{NIF} TO f_m

IF Noise Bandwidth equivalent f_m	Discriminator Input SNR at Thresholds
2	6 db
5	8 db
8	10 db
15	12 db
25	13 db

APPENDIX F

FEEDBACK DISCRIMINATOR DESIGN

F.1 GENERAL

This appendix is devoted to development of the equations used in designing two-pole feedback discriminators. The first of these equations to be derived is the equation relating the effective β in the IF portion of the loop, denoted β_{IF}, with the feedback factor F and the input β, defined as β_{RF}.

The signal applied to the IF is simply the output of the mixer which is E(s) in a linearized loop (See Fig. A-4 in Appendix A). Now one finds the amount (t) changes, $\epsilon(t) = \mathcal{L}^{-1}$ E(s), for a given change in the input frequency. This has already been determined in equation (C-13) of Appendix C, and since only the steady state value is of interest:

$$\epsilon(t)_{ss} = f_d \sqrt{\frac{4\zeta^2\omega_n^2\omega_m^2 + (ab - \omega_m^2)^2}{4\zeta^2\omega_n^2\omega_m^2 + (\omega_n^2 - \omega_m^2)^2}} \qquad \text{(F-1)}$$

Defining F so that $\omega_n^2 = N\, b^2 F$, Equation (F-1) becomes for $F^2 \gg 1$:

$$\epsilon(t)_{ss} \approx \frac{f_d}{F} \sqrt{\frac{2(N^2+1)}{N^2}} \qquad \text{(F-2)}$$

But the IF frequency shift is $\epsilon(t)_{ss}$ or $\Delta f_{IF} = \epsilon(t)_{ss}$ while the RF frequency shift is $\Delta f_{RF}\ f_x$

then
$$\Delta f_{IF} = \frac{\Delta f_{RF}}{F} \sqrt{\frac{2(N^2+1)}{N^2}}$$

hence
$$\beta_{IF} = \frac{\beta_{RF}}{F} \sqrt{\frac{2(N^2+1)}{N^2}}$$

or
$$F = \frac{\beta_{RF}}{\beta_{IF}} \sqrt{\frac{2(N^2+1)}{N^2}} \qquad \text{(F-3)}$$

Equation (F-3) expresses the desired relation between F, β_{RF} and β_{IF}. The next relation that will be obtained is the connection between F and the ratio of IF to low pass bandwidth. N is defined to be this bandwidth ratio. Before proceeding with the derivations, the following definitions will be restated:

$$a + b \triangleq 2\zeta\omega_n$$

$$a\,b\,F \triangleq \omega_n^2$$

Substituting $a = Nb$:

$$b\,(1 + N) = 2\zeta\omega_n \tag{F-4}$$

and

$$b^2\,NF = \omega_n^2 \tag{F-5}$$

Solving equations (F-4) and (F-5) for N yields:

$$N = 2\zeta^2 F - 1 + 2\zeta\sqrt{\zeta^2 F^2 - F} \tag{F-6}$$

Moreover, in order for the closed loop threshold to predominate (See Appendix E), the closed loop noise bandwidth must exceed the IF noise bandwidth by some factor. The value of this factor is determined by F, B_{nl}, and B_{NIF}; but B_{nl} is in turn a function of N, completing the circle. If $b = \omega_m$, then $N = \beta_{IF}$. If the open loop threshold requirement is not met, then either the chosen ratio between closed noise bandwidth and IF noise bandwidth must be increased or a larger F, which in turn reduces B_{IF}, must bc used.

Assuming $B_{nl} \geqq B_{NIF}$

$$\frac{\omega_n}{4\zeta} \geqq B_{NIF}$$

$B_{n1} = \frac{\omega_n}{4\zeta}$ from equation (D-12) in Appendix D. The noise bandwidth of a single pole filter is about 1.57 times the 3 db bandwidth. One-half the IF 3 db bandwidth is equal to a [See Equation (A-13)] which in turn is equal to Nb, and therefore one-half the IF db bandwidth is:

$$a = Nb$$

$$= \omega_n \sqrt{\frac{N}{F}}$$

Then the IF noise bandwidth, considering a two-sided noise spectrum is:

$$B_{NIF} = \frac{\pi}{2\pi} \quad \omega_n \sqrt{N/F} \qquad \text{(F-7)}$$

Substituting the right side of equation (F-7) into equation (F-6) gives:

$$\frac{\omega_n}{4\zeta} \geq \frac{1.57}{\pi} \quad \omega_n \sqrt{N/F} \qquad \text{(F-8)}$$

Solving equation (F-8) for ζ :

$$\zeta \leq 1/2 \sqrt{F/N}$$

Now letting $\zeta \leq 1/2\sqrt{F/N}$ in equation (F-6) gives the result that:

$$N \leq F - 1 \qquad \text{(F-9)}$$

Equation (F-8) expresses N as a function of F for the case when closed loop noise bandwidth is equal to or greater than the IF noise bandwidth.

$$N \leq \frac{F}{2} - 1 \text{ holds for } B_{N1} \geq 2B_{NIF}\text{, and } \zeta \leq \frac{1}{4}\sqrt{\frac{F}{N}}$$

The IF bandwidth is determined by the modulation frequency and B_{IF} . Assuming sinusoidal modulation, the IF time varying voltage is:

$$e_{IF}(t) = A \sin (\omega_o t + B_{IF} \sin \omega_{mt}) \qquad \text{(F-10)}$$

where:

e_{IF} = IF voltage

A = peak IF voltage

ω_o = center IF frequency

Equation (F-11) can be expanded using the relations sin (A + B) = sin A cos B + cos A sin B into:

$$e_{IF}(t) = A \left[\sin \omega_o t \cos B_{IF} (\sin \omega_{mt}) + \cos \omega_o t \sin B_{IF} (\sin \omega_{mt})\right] \qquad \text{(F-11)}$$

but:

$$\cos B_{IF} (\sin \omega_{mt}) = J_o (B_{IF}) + 2 J_2 (B_{IF}) \cos 2 \omega_{mt}$$

$$+ 2 J_4 (B_{IF}) \cos 4 \omega_{mt} + \text{ and } \ldots\ldots$$

$$\sin B_{IF} (\sin \omega_{mt}) = 2 J_1 (B_{IF}) \sin \omega_{mt}$$

$$+ 2 J_3 (B_{IF}) \sin 3 \omega_{mt} + \ldots\ldots\ldots$$

$J_n (B_{IF})$ represents a Bessel function of the first kind and order n with argument B_{IF} . Therefore, equation (F-11) becomes:

$$e_{IF}(t) = A \Big\{ J_o (B_{IF}) \sin \omega_o t$$

$$+ J_1 (B_{IF}) \Big[\sin (\omega_o t + \omega_m) t - \sin (\omega_o - \omega_m) t\Big]$$

$$+ J_2 (B_{IF}) \Big[\sin (\omega_o + 2 \omega_m) t + \sin (\omega_o - 2 \omega_{mt})\Big]$$

$$+ \ldots\ldots\ldots\ldots \Big\} \qquad \text{(F-12)}$$

In equation (F-12), $J_o (B_{IF})$ is the carrier amplitude and $J_n (B_{IF})$ is the amplitude of the n^{th} sideband. In order to determine the required IF bandwidth for a given B_{IF}, the Bessel functions in equation (F-10) are evaluated for that B_{IF} , and only the sidebands having significant amplitude need be passed by the IF filter. For example, if $B_{IF} = 2$:

$$J_o (2) = 0.2239$$

$$J_1 (2) = 0.5767$$

$$J_2 (2) = 0.3528$$

$J_3(2) = 0.1289$

$J_4(2) = 0.03400$, etc. (Ref. 7)

Hence, when $B_{IF} = 2$, sidebands whose order exceeds 3 can be neglected because their amplitudes are small, and $B_{IF} = 6 f_m$.

In Appendix F, an expression relating F, B_{RF} and B_{IF} is embodied in equation (F-3), and equation (F-7) relates N to F. Additionally, the method of determining B_{IF} has been derived.

APPENDIX G

SNR PROPERTIES OF STANDARD AND FEEDBACK DISCRIMINATORS

As the first step in examining the SNR characteristics, the determination of the bandwidth required by any given β will be found. Equation (F-10) of Appendix F is generalized by letting $\beta = \beta_{IF}$, $\omega_c = \omega_o$, and $e = e_{IF}$. When this is done:

$$e = A \Big\{ J_o(\beta) \sin \omega t + J_1(\beta) \left[\sin(\omega_c + \omega_m) t \right] + J_2(\beta) \left[\sin(\omega_c + 2\,\omega_m) t + \sin(\omega_c - \omega_m t) \right] + \ldots\ldots \Big\} \qquad \text{(G-1)}$$

Equation (G-1) is evaluated (Ref. 7) for various values of β, and the sidebands whose Bessel function co-efficients are less than 0.03 are disgarded; the result is shown in Figure G-1 where required bandwidth divided by f_m is plotted as a function of β. Figure G-1 can be used to find the required information bandwidth for a given value of β, if β is between zero and 20. Moreover, the information bandwidth found from Figure G-1 is also the noise bandwidth, and this noise bandwidth will be used in determining the signal to noise ratio improvement resulting from the use of frequency feedback.

The output SNR for a standard discriminator is derived by Schwartz (Ref. 5) and is:

$$\rho_o = \frac{3\,\beta^2 \left(\beta_{NRF} \times 2\right)}{2f_m} \quad \rho_1 \qquad \text{(G-2)}$$

where

ρ_o = output SNR

ρ_1 = input SNR

β_{NRF} = input noise bandwidth (low pass equivalent)

The value of β_{NIF} in Equation (G-2) is equal to the required information bandwidth, assuming ideal square filtering.

Equation (G-2) expresses the output SNR for a standard discriminator, and now an equation expressing the output signal to noise ratio of a feedback discriminator will be found. Assuming the time varying voltage (no noise) applied to the loop is:

$$e_s\ (t) = A_c \sin \left(\omega_c\, t + \frac{f_d}{f_m} \cos \omega_m t\right) ,$$

the equivalent frequency input (see Appendix A) is:

$$f = f_d \sin \omega_m t \qquad \text{(G-3)}$$

Taking the Laplace transform of equation (G-3) yields:

$$F_1\ (s) = \frac{f_d\, \omega_m}{S^2 + \omega_m^2} \qquad \text{(G-4)}$$

The loop transfer function at the low pass filter output is obtained from Appendix B, and it is:

$$\frac{C(s)}{R(s)} = \frac{K_d K_a ab}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad \text{(G-5)}$$

where C(s) and R(s) are defined in Figure A-4 of Appendix A.

R(s) is the equivalent frequency input which is equal to F(s), (S) in this case. C(s) is taken to be the output signal. Therefore,

$$C(s) = \frac{f_d \omega_m}{S^2 + \omega_m^2} \cdot \frac{K_d K_a ab}{S^2 + 2\zeta\omega_n S + \omega_n^2} \qquad \text{(G-6)}$$

The steady state output voltage (in the time domain) is obtained by taking the inverse Laplace transform of equation (G-6),

$$\text{or} \quad e_o = \frac{f_d K_A K_d ab}{\sqrt{4\zeta^2 \omega_n^2 \omega_m^2 + \left(\omega_n^2 - \omega_m^2\right)^2}} \sin(\omega_m t + \psi) \qquad \text{(G-7)}$$

But since $\beta_{RF} = \frac{f_d}{f_m}$, equation (G-7) becomes:

$$\rho_o = \frac{\beta_{RF} f_m K_a K_d ab \sin\left(\omega_m t + \psi\right)}{\sqrt{4\zeta^2 \omega_n^2 \omega_m^2 + \left(\omega_n^2 - \omega_m^2\right)^2}} \qquad \text{(G-8)}$$

The normalized output signal power (based on a 1 Ω load) is simply the square of the rms value of ρ_o in equation (G-8); signal power is:

$$P_S \text{ out} = \frac{\beta_{RF}^2 f_m^2 K_A^2 K_d^2 a^2 b^2}{4\zeta^2 \omega_n^2 \omega_m^2 + \left(\omega_n^2 - \omega_m^2\right)^2} \qquad \text{(G-9)}$$

Now that the signal power has been determined, it is only necessary to derive the output noise power in order to find output SNR. The

technique used to ascertain available output noise power is finding the equivalent input noise power spectral density and multiplying it by the closed loop noise bandwidth. Of course, this approach is predicted on the assumption that the input noise has a flat spectrum.

Recalling from Appendix A, that the input SNR is:

$$\rho_i = \frac{A_c^2}{2 N_o \beta_{NRF}} \qquad \text{(G-10)}$$

Where

A_c = carrier amplitude

N_o = noise power spectral density

Equation (G-10) is normalized by dividing numerator and denominator of the right side of A_c^2 so that it becomes:

$$\rho_i = \frac{1}{2 N_o' \beta_{NRF}} \qquad \text{(G-11)}$$

where

$N_o' = N_o / A_c^2$

and

N_o' is in $(\text{cps})^{-1}$

N_o' in Equation (G-11) is the normalized noise power spectral density. From Appendix A, the equivalent noise power spectral density for a frequency input is:

$$G_f(f) = \frac{f^2 N_o}{A_c^2} \qquad \text{or} \qquad G_f(\omega) = \frac{\omega^2}{4\pi^2} N_o'$$

but:

$$N_o' = \frac{1}{i^2 \beta_{NRF}} ,$$

Hence,

$$G_f(\omega) = \frac{\omega^2}{8\pi^2 \rho_i \beta_{NRF}} .$$

If a rectangular post-detection filter having a bandwidth of f_m filters the loop output, then the output noise power is:

$$P_N = \frac{1}{8\pi^2 \rho_i \beta_{NRF}} \int_{-\omega_m}^{\omega_m} \left| \frac{C(j\omega)}{R(j\omega)} \right|^2 \omega^2 \frac{d\omega}{2\pi} \tag{G-12}$$

but:

$$\left| \frac{C(j\omega)}{R(j\omega)} \right|^2 = \frac{a^2 b^2 K_d^2 K_A^2}{\left(\omega_n^2 - \omega^2\right)^2 + 4\zeta \omega_n^2 \omega^2}$$

When the above substitution for $\left| \frac{C(j\omega)}{R(j\omega)} \right|^2$ is inserted into equation (G-12), evaluation of the integral is exceedingly difficult. However, if a post-detection filter having a finite number of poles is used, the integration limits are $-\infty$ to $+\infty$, and then the integral is found by contour integration as was done in Appendix D. Using this method, P_N is:

$$P_N = \frac{a^2 b^2 K_A^2 K_d^2 \omega_m^3}{12\pi^3 B_{NRF} \rho_i \omega_n^4} \tag{G-13}$$

The output SNR of a feedback discriminator is found by dividing equation (G-9), which is output signal power for sinusoidal modulation, by equation (G-13). The result is:

$$\rho_o = \frac{12\pi^3 \left(\beta_{RF} f_m\right)^2 \omega_n^4 \beta_{NRF}}{\omega_m^3 \left[4\zeta^2 \omega_n^2 \omega_m^2 + \left(\omega_n^2 - \omega_m^2\right)^2\right]} \rho_i \qquad \text{(G-14)}$$

Equation (G-14) is simplified if the assumption that $\omega_n \gg \omega_m$ is made, and this assumption is generally valid since N is constrained so that,

$$N \leq \frac{F}{2} - 1 \quad \text{(See equation (G-7) of Appendix F.)}$$

With $\omega_n \gg \omega_m$, equation (G-14) becomes:

$$\rho_o \approx \frac{3 \left(\beta_{RF} f_m\right)^2 \quad 4\pi^3}{\omega_m^3} \beta_{NRF} \; \rho_i \qquad \text{(G-15)}$$

But $\omega_m^2 = 8\pi^3 f_m^3$, so equation (G-15) reduces to:

$$\rho_o \approx \frac{3\beta_{RF}^2}{2} \left(\frac{\beta_{NRF}}{f_m}\right) \rho_i \qquad \text{(G-16)}$$

In Appendix G, three main aspects of SNR properties have been considered. Firstly, Figure G-1 is a plot of required open loop bandwidth for values of β between zero and 20. Equation (G-2) expresses output SNR of a standard discriminator; Equation (G-16) the output SNR of a feedback discriminator with post-detection filtering.

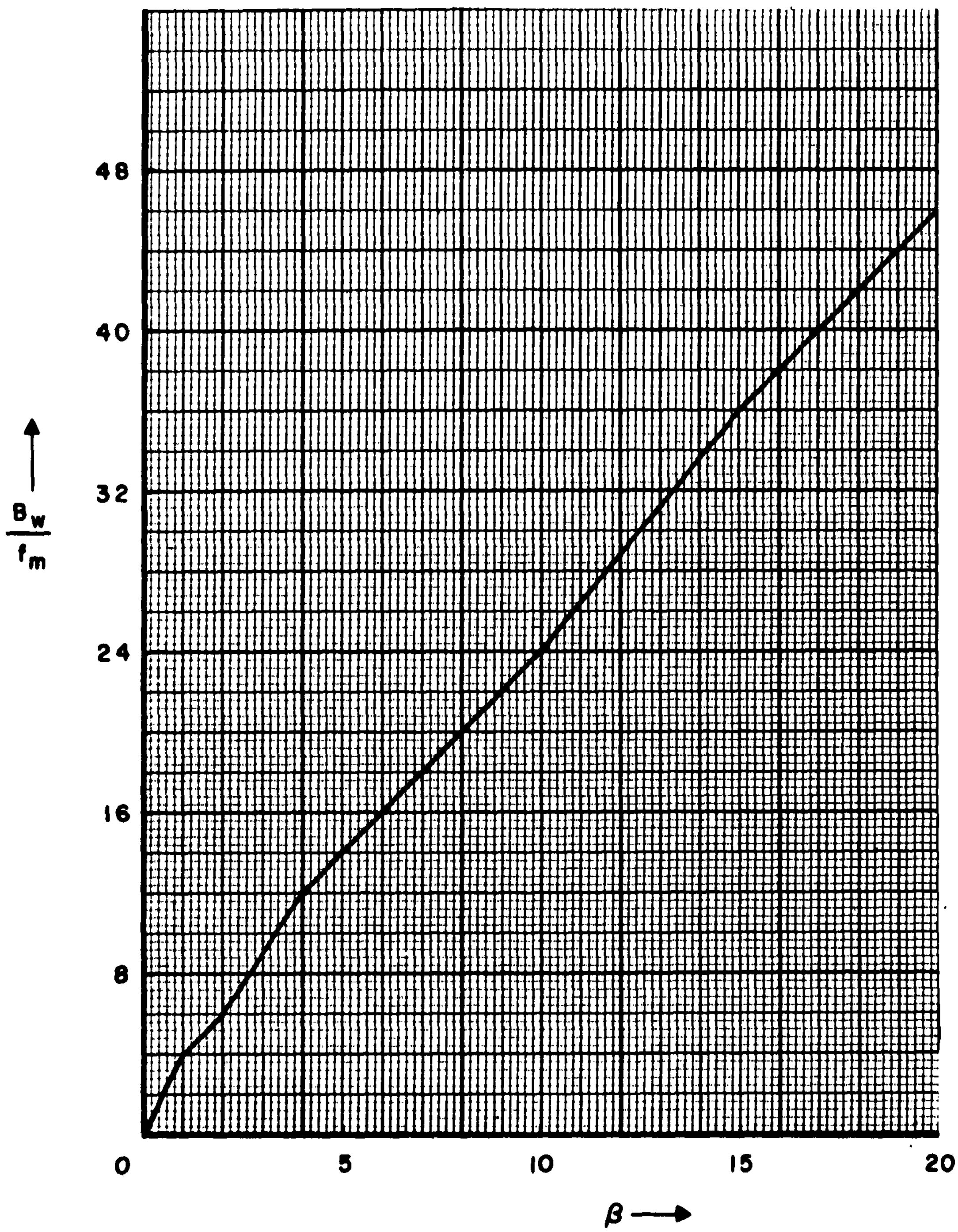


Fig. G-1 Required Bandwidth Divided by f_m vs β

APPENDIX H
EXPERIMENTAL RESULTS

H. 1 GENERAL

The objectives of the experimental work were to verify that a feedback discriminator does indeed have a frequency compression property and that it has a lower threshold than a conventional discriminator. In addition, the damping factor, ρ, and undamped natural angular frequency, W_{nl} were measured and compared with calculated theoretical parameter values.

A spectrum analyzer verified that a feedback discriminator does compress its input frequency spectrum.

Enloe's and Chaffee's closed loop threshold criteria were used; i.e., threshold occurs when one hears a "pop" approximately every second through earphones attached to the loop output. Thus, a closed loop threshold SNR of approximately 6 db in B_{nl}, closed loop noise bandwidth, was measured. A standard discriminator would have had, approximately, a 10 db threshold ($B_{nl} \approx B_{NRF}$ in this experiment). The damped factor's measured value was 0.37, in close agreement with the calculated value of 0.376. The undamped natural frequency, $W_n/2\pi$ measured 23 kc, compared with a calculated value of 20 kc. Relative closed loop output as a function of angular frequency is plotted on Fig. H-1.

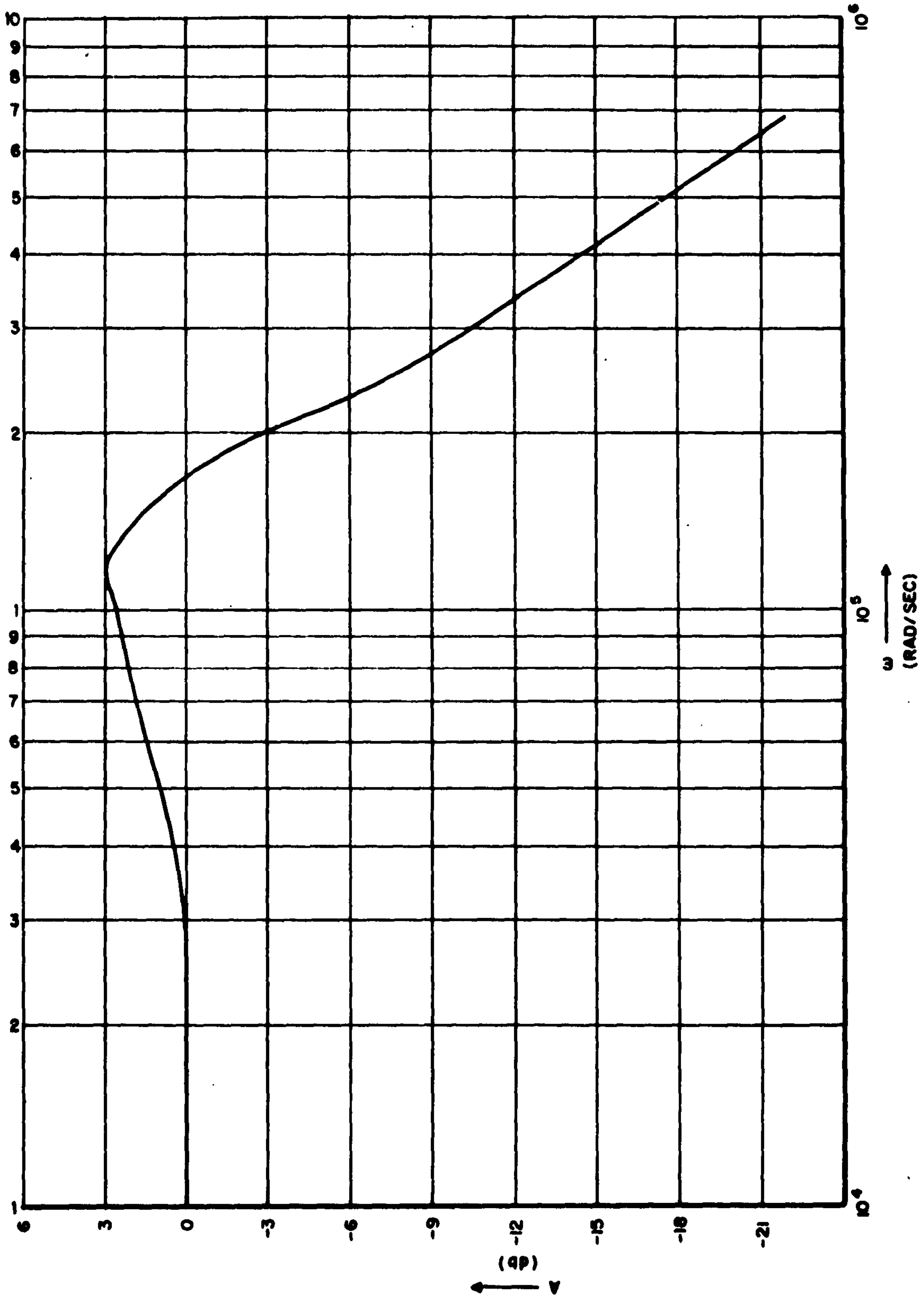


Fig. H-1 Closed Loop Output Amplitude vs. Angular Frequency

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